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INTRODUCTION

The theory of distributions is a generalization of classical analysis, which makes it possible to deal in a systematic manner with difficulties which previously had been overcome by *ad hoc* constructions, or just by pure hand waving. In fact, it does a good deal more: it provides a new and wider framework, and a more perspicuous language, in which one can reformulate and develop classical problems. Its influence has been particularly pervasive and fruitful in the theory of linear partial differential equations.

Let us consider some examples. If $(x, t) \in \mathbf{R}^2$, then

$$u = f(x+t) + g(x-t)$$

satisfies d'Alembert's equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0,$$

provided that the functions f and g are twice differentiable. This restriction is both irksome and unnatural in many instances. It can be overcome by introducing so-called weak solutions. By definition, these are functions u such that

$$\int u \left(\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} \right) \mathrm{d}x \, \mathrm{d}t = 0$$

for all sufficiently 'good' functions ϕ, for example for $\phi \in C_c^2(\mathbf{R}^2)$, the class of twice continuously differentiable functions that vanish on the exterior of a bounded set.

Again, if $x \in \mathbf{R}^3$, then the Newtonian potential

$$u(x) = \int \frac{f(y)}{|x-y|} \, \mathrm{d}y$$

satisfies Poisson's equation

$$\Delta u = -4\pi f$$

if the density function f is, for example, continuously differentiable. But it may not possess second order derivatives when f is merely continuous. Yet, it is then continuously differentiable and obeys Gauss's law that the flux of the field across a closed surface S is proportional to the matter enclosed by S. This difficulty can also be avoided by working with weak solutions of Poisson's equation. Furthermore, if one replaces f by the Dirac delta 'function', one obtains $u = 1/|x| = (x_1^2 + x_2^2 + x_3^2)^{-1/2}$ when $x \neq 0$, which is the potential due to a particle at the origin. It is suggestive to express this by writing

$$\Delta(1/|x|) = -4\pi\delta(x).$$

But this has to be interpreted; for example, one can take it to mean that

$$\int \frac{\Delta\phi(x)}{|x|}\, dx = -4\pi\phi(0)$$

for all 'good' functions, say for all $\phi \in C_c^2(\mathbf{R}^3)$.

In all these cases, the difficulties and ambiguities disappear when the equations are read in terms of distributions. In the theory of distributions, functions are replaced by linear forms on an auxiliary vector space, whose members are called test functions. Recall that, if V is a vector space over the field $\mathbf{C}$ of complex numbers, then a linear form on V is a homomorphism $V \to \mathbf{C}$. The linear forms on V are made into a vector space $\mathrm{Hom}(V, \mathbf{C})$ in the obvious way: $\langle cu, \phi\rangle = c\langle u, \phi\rangle$ if $c \in \mathbf{C}$ and $u \in \mathrm{Hom}(V, \mathbf{C})$, and $\langle u + v, \phi\rangle = \langle u, \phi\rangle + \langle v, \phi\rangle$ if $u, v \in \mathrm{Hom}(V, \mathbf{C})$, where $\phi \in V$ in each case. In distribution theory, the basic space of test functions is $C_c^\infty(\mathbf{R}^n)$; its members are (complex valued) functions on $\mathbf{R}^n$ which possess continuous derivatives of all orders, and vanish outside some bounded set. The notations $C_0^\infty(\mathbf{R}^n)$, and L. Schwartz's original $\mathscr{D}(\mathbf{R}^n)$, are also used.

A continuous function f: $\mathbf{R}^n \to \mathbf{C}$ determines a linear form on $C_c^\infty(\mathbf{R}^n)$ by the rule

$$\langle f, \phi\rangle = \int_{\mathbf{R}^n} f\phi\, dx, \quad \phi \in C_c^\infty(\mathbf{R}^n). \tag{1}$$

Conversely, it can (and will) be shown that this linear form determines f uniquely so that the space of continuous functions on $\mathbf{R}^n$ can be identified with a subspace of $\mathrm{Hom}(C_c^\infty(\mathbf{R}^n), \mathbf{C})$. If the function f is also continuously differentiable, then the linear forms on $C_c^\infty(\mathbf{R}^n)$ determined by its derivatives are, by (1) and an integration by parts,

$$\langle \partial f/\partial x_i, \phi\rangle = \int \phi(\partial f/\partial x_i)\, dx = -\int f(\partial\phi/\partial x_i)\, dx,$$
$$i = 1, \ldots, n,\ \phi \in C_c^\infty(\mathbf{R}^n).$$

Thus, for $i = 1, \ldots, n$,

$$\langle \partial f/\partial x_i, \phi\rangle = -\langle f, \partial\phi/\partial x_i\rangle, \quad \phi \in C_c^\infty(\mathbf{R}^n). \tag{2}$$

But this makes sense for any linear form on $C_c^\infty(\mathbf{R}^n)$, and so provides a definition of the derivatives of such a form. As one can iterate (2), one thus obtains (generalized) derivatives of all orders.

Multiplication by smooth (infinitely differentiable) functions can also be defined by analogy with the special case (1); one simply puts

$$\langle fu, \phi\rangle = \langle u, f\phi\rangle, \quad \phi \in C_c^\infty(\mathbf{R}^n). \tag{3}$$

By combining (2) and (3), one can thus account for the action of any linear differential operator with smooth coefficients on $\mathrm{Hom}(C_c^\infty(\mathbf{R}^n), \mathbf{C})$.

There is one other essential ingredient. The class of distributions is not the whole of $\mathrm{Hom}(C_c^\infty(\mathbf{R}^n), \mathbf{C})$: it is the subspace consisting of continuous linear forms. To say this, presupposes that $C_c^\infty(\mathbf{R}^n)$ has been equipped with an appropriate topology. The choice of this topology is, in fact, a cardinal feature of the theory of distributions. To define it, and to explore its implications, one must appeal to the theory of locally convex topological vector spaces. However, the course adopted in this book is to specify a certain set of inequalities which a linear form on $C_c^\infty(\mathbf{R}^n)$ must satisfy in order to qualify as a distribution. Once these are granted, the theory can be built up systematically and logically. But, so as to give some idea of what is involved, to readers who either do not have the time, or the inclination, to go into this more fully, a sketch of the functional-analytic background is given as an Appendix at the end of the book. This can be omitted, and the reader who does so should also ignore references to topological vector spaces in the text.

1 TEST FUNCTIONS AND DISTRIBUTIONS

1.1. Some notations and definitions

Throughout this book, the letter $\mathbf{R}$ will denote both the field of real numbers and the real line, and the letter $\mathbf{C}$ will stand for both the field of complex numbers and the complex plane. $\mathbf{R}^n$ will be understood to have its usual vector space structure and inner product, so that

$$cx = (cx_1, \ldots, cx_n) \quad \text{if} \quad c \in \mathbf{R} \quad \text{and} \quad x = (x_1, \ldots, x_n) \in \mathbf{R}^n,$$

and

$$x + y = (x_1 + y_1, \ldots, x_n + y_n), \quad x \cdot y = \sum_{j=1}^{n} x_j y_j$$

if $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$. The Euclidean norm $(x \cdot x)^{1/2}$ will be written as $|x|$. We always write $\mathbf{R}$ instead of $\mathbf{R}^1$.

If A is a subset of $\mathbf{R}^n$, then $\bar{A}$ is its closure and ∂A is its boundary. It will be recalled that $A \subset \mathbf{R}^n$ is compact if and only if it is both closed and bounded; if $X \subset \mathbf{R}^n$ is an open set then A is a compact subset of X if it is compact and $A \subset X$. If A and B are subsets of $\mathbf{R}^n$, we shall write $A \setminus B$ for the (set-theoretic) difference $\{x : x \in A, x \notin B\}$.

If X and Y are sets, and f is a function on X with range in Y, one writes $f: X \to Y$, and $x \mapsto y$ or $x \mapsto f(x)$ indicates that f maps $x \in X$ to $y = f(x) \in Y$. A map is injective if $f(x') = f(x'')$ implies that $x' = x''$, surjective if every $y \in Y$ is the image of some $x \in X$, and bijective if it is both injective and surjective.

Let $X \subset \mathbf{R}^n$ be an open set, and let k be a nonnegative integer. The class $C^k(X)$ consists of the complex valued functions on X which have continuous derivatives of order less than, or equal to, k. (The function itself is included, conventionally, as the 'derivative of order zero'.) Likewise, $C^\infty(X)$ consists of the functions which have continuous derivatives of all orders.

The *support* of a function $f: X \to \mathbf{C}$ is the closure of the set $\{x \in X : f(x) \neq 0\}$; note that it is a closed subset of X. We shall write $\operatorname{supp} f$ for this. Functions with compact support play an important part in the theory. We write $C_c^k(X)$ for the subset of $C^k(X)$ consisting of functions with compact support, and

$C_c^\infty(X)$ for the subset of $C^\infty(X)$ consisting of functions with compact support. Note that $C^k(X)$, $C^\infty(X)$, $C_c^k(X)$ and $C_c^\infty(X)$ are all vector spaces over **C**.

Integrals are Lebesgue integrals. When dealing with functions defined on a fixed open subset X of $\mathbf{R}^n$, we omit the domain of integration, so that

$$\int f(x)\,\mathrm{d}x = \int_X f(x)\,\mathrm{d}x$$

by definition; here, dx is Lebesgue measure.

Let $X \subset \mathbf{R}^n$ be an open set, and $f \in C^\infty(X)$. We shall usually write the derivatives as

$$\partial_j f = \partial f/\partial x_j, \quad j = 1, \ldots, n. \tag{1.1.1}$$

Derivatives of higher order can be written concisely by means of the multi-index notation. A *multi-index* (or, to be precise, an n-multi-index) is an n-tuplet $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers; its length (or order) is $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The sum of two multi-indices α and β is $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$. One says that $\beta \leqslant \alpha$ if $\beta_j \leqslant \alpha_j$ for $j = 1, \ldots, n$; when $\beta \leqslant \alpha$ one can also define $\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n)$. One now sets

$$\partial^\alpha f = \partial^{|\alpha|} f/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}, \tag{1.1.2}$$

so that

$$\partial^\alpha \partial^\beta f = \partial^{\alpha+\beta} f. \tag{1.1.3}$$

This obviously also applies to $f \in C^k(X)$, provided that $|\alpha + \beta| \leqslant k$.

To complete the multi-index notation, we put

$$\alpha! = \alpha_1! \ldots \alpha_n! \tag{1.1.4}$$

for any multi-index α, and if also $x \in \mathbf{R}^n$ we set

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}. \tag{1.1.5}$$

The formal statement of Taylor's theorem then becomes

$$f(x + h) = \sum_{\alpha \geqslant 0} \frac{x^\alpha}{\alpha!} \partial^\alpha f(h), \tag{1.1.6}$$

and the multinomial theorem assumes the concise form

$$(x_1 + \cdots + x_n)^m = \sum_{|\alpha| = m} \frac{m!}{\alpha!} x^\alpha. \tag{1.1.7}$$

1.2. Test functions

Let $X \subset \mathbf{R}^n$ be an open set. The members of the vector space $C_c^\infty(X)$ are called test functions. To give an example, we note first that, if $t \in \mathbf{R}$, the function

$$\psi_0(t) = e^{1/t} \quad \text{if} \quad t < 0, \quad \psi_0(t) = 0 \quad \text{if} \quad t \geqslant 0 \tag{1.2.1}$$

is a member of $C^\infty(\mathbf{R})$; the easy proof is left to the reader. Now put

$$\psi(x) = \psi_0(|x|^2 - 1). \tag{1.2.2}$$

Then $\psi \in C_c^\infty(\mathbf{R}^n)$, and the support of ψ is the closed unit ball $\{|x| \leqslant 1\}$. Other test functions can then be constructed by *regularization*; this is a method of 'smoothing' that will be extended to distributions in Chapter 4. For the present, we prove:

Theorem 1.2.1. Let $f \in C_c^k(\mathbf{R}^n)$, where $0 \leqslant k < \infty$. Let $\rho \in C_c^\infty(\mathbf{R}^n)$ be such that

$$\rho \geqslant 0, \quad \operatorname{supp} \rho \subset \{|x| \leqslant 1\}, \quad \int \rho \, dx = 1, \tag{1.2.3}$$

let ϵ be a positive real number, and put

$$f_\epsilon(x) = \epsilon^{-n} \int f(y) \rho\left(\frac{x - y}{\epsilon}\right) dy. \tag{1.2.4}$$

Then $f_\epsilon \in C_c^\infty(\mathbf{R}^n)$, and the support of f_ϵ is contained in the ϵ-neighbourhood of the support of f; moreover, if $|\alpha| \leqslant k$, then the $\partial^\alpha f_\epsilon$ converge uniformly to $\partial^\alpha f$ as $\epsilon \to 0$.

Note. One can, for example, take $\rho = \psi / \int \psi \, dx$ where ψ is given by (1.2.2).

Proof. The first statement is obvious, since repeated differentiation under the integral sign is permissible, and $f_\epsilon = 0$ when the distance of x from $\operatorname{supp} f$ exceeds ϵ. To prove the second assertion, write (1.2.4) as

$$f_\epsilon(x) = \int f(x - \epsilon z) \rho(z) \, dz. \tag{1.2.5}$$

By (1.2.3) one then has

$$|f_\epsilon(x) - f(x)| = \left| \int (f(x - \epsilon z) - f(x)) \rho(z) \, dz \right|$$

$$\leqslant \int |f(x - \epsilon z) - f(x)| \, \rho(z) \, dz$$

whence, again by (1.2.3),

$$|f_\epsilon(x) - f(x)| \leqslant \sup \{|f(x + y) - f(x)| : |y| \leqslant \epsilon\}.$$

This tends to zero uniformly as $\epsilon \to 0$, by the uniform continuity of f. If $|\alpha| \leqslant k$, one can differentiate under the integral sign in (1.2.5) to obtain

$$\partial^\alpha f_\epsilon(x) = \int \partial^\alpha f(x - \epsilon z) \rho(z) \, dz$$

and the same argument shows that $\partial^\alpha f_\epsilon \to \partial^\alpha f$ as $\epsilon \to 0$, with uniform convergence, and so we are done.

Remark. If $X \subset \mathbf{R}^n$ is an open set and $f \in C_c^k(X)$, then one can extend f to $\mathbf{R}^n$ by setting $f=0$ on $\mathbf{R}^n \setminus \bar{X}$. Since $f_\epsilon \in C_c^\infty(X)$ when $\epsilon < \inf\{|x-y| : x \in \operatorname{supp} f\}$, $y \in \mathbf{R}^n \setminus X$ it is apparent that f can be approximated by test functions. Indeed, one can construct in this way a sequence $(f_j)_{1 \leqslant j < \infty} \in C_c^\infty(X)$ such that $\partial^\alpha f_j \to \partial^\alpha f$ as $j \to \infty$ if $|\alpha| \leqslant k$, the convergence being uniform.

The hypothesis that $\rho \geqslant 0$ will be useful in some applications but can obviously be dropped without invalidating Theorem 1.2.1.

1.3. Distributions

We are now ready to introduce distributions.

Definition 1.3.1. Let $X \subset \mathbf{R}^n$ be an open set. A linear form $u: C_c^\infty(X) \to \mathbf{C}$ is called a distribution if, for every compact set $K \subset X$, there is a real number $C \geqslant 0$ and a nonnegative integer N such that

$$|\langle u, \phi \rangle| \leqslant C \sum_{|\alpha| \leqslant N} \sup |\partial^\alpha \phi|, \tag{1.3.1}$$

for all $\phi \in C_c^\infty(X)$ with $\operatorname{supp} \phi \subset K$. The vector space of distributions on X is called $\mathscr{D}'(X)$.

Note. Inequalities such as (1.3.1) are called *semi-norm estimates*; the reason for this is explained in the Appendix.

The vector space structure of $\mathscr{D}'(X)$ is defined in the obvious way:

$$\langle u+v, \phi \rangle = \langle u, \phi \rangle + \langle v, \phi \rangle, \quad u, v \in \mathscr{D}'(X), \quad \phi \in C_c^\infty(X),$$
$$\langle cu, \phi \rangle = c\langle u, \phi \rangle, \quad c \in \mathbf{C}, \quad u \in \mathscr{D}'(X), \quad \phi \in C_c^\infty(X).$$

Our first example is simple, but important.

Theorem 1.3.1. Let $X \subset \mathbf{R}^n$ be an open set, and let $f \in C^0(X)$. Then

$$\langle f, \phi \rangle = \int f\phi \, \mathrm{d}x, \quad \phi \in C_c^\infty(X) \tag{1.3.2}$$

is a distribution. Furthermore, if the second member of (1.3.2) vanishes for all $\phi \in C_c^\infty(X)$, then $f=0$ on X.

Proof. That (1.3.2) is a distribution is obvious, since

$$\left| \int f\phi \, \mathrm{d}x \right| \leqslant \sup |\phi| \int_K |f| \, \mathrm{d}x, \quad \phi \in C_c^\infty(K), \tag{1.3.3}$$

where $K \subset X$ is a compact set and

$$C_c^\infty(K) = \{\phi : \phi \in C_c^\infty(\mathbf{R}^n), \quad \operatorname{supp} \phi \subset K\}. \tag{1.3.4}$$

Suppose now that the second member of (1.3.2) vanishes for all $\phi \in C_c^\infty(X)$, but that $f(y) \neq 0$ at some point $y \in X$. By the continuity of f, there is then a $\delta > 0$ such that $\operatorname{Re}(f(x)/f(y)) \geqslant \frac{1}{2}$ if $|x-y| \leqslant \delta$, and one can clearly assume that $\{x: |x-y| \leqslant \delta\} \subset X$. Hence, with a real valued $\rho \in C_c^\infty(\mathbf{R}^n)$ satisfying (1.2.3), one has

$$0 = \operatorname{Re} \int \frac{f(x)}{f(y)} \rho\left(\frac{x-y}{\delta}\right) \mathrm{d}x \geqslant \tfrac{1}{2} \int \rho\left(\frac{x-y}{\delta}\right) \mathrm{d}x = \tfrac{1}{2}\delta^n,$$

which is absurd. So the theorem is proved.

Thus, (1.3.2) gives an injective map $C^0(X) \to \mathscr{D}'(X)$, and one can identify a continuous function f with the distribution defined in this way. We shall usually speak of the distribution determined by f, or of the distribution equal to f.

Remark. It is obvious from (1.3.3) that (1.3.2) also yields a distribution on X when f is *locally integrable*, that is to say when (it is measurable and) $\int_K |f| \, \mathrm{d}x < \infty$ for all compact sets $K \subset X$. This distribution cannot determine f uniquely, as the second member of (1.3.2) is unchanged when f is replaced by a function g that is equal to f almost everywhere. (This means that $\{x \in X: f(x) - g(x) \neq 0\}$ is a set of measure zero. A set of measure zero is one that can, given any $\epsilon > 0$, be covered by a countable family of rectangles whose total volume is less than ϵ.) However, it is not difficult to show that, if f is locally integrable, and the second member of (1.3.2) vanishes for all $\phi \in C_c^\infty(X)$, then $f = 0$ almost everywhere. (The reader familiar with integration will be able to deduce this from Theorem 1.2.1 by considering the measure $f\,\mathrm{d}x$.) By definition, the vector space $L_1^{\mathrm{loc}}(X)$ consists of all equivalence classes of locally integrable functions on X. So (1.3.2) gives an injective map $L_1^{\mathrm{loc}}(X) \to \mathscr{D}'(X)$; we shall simply refer to the distribution determined by (or 'equal to') f.

Another basic example is the *Dirac distribution* (delta function) which is the member of $\mathscr{D}'(\mathbf{R}^n)$ given by

$$\langle \delta, \phi \rangle = \phi(0), \quad \phi \in C_c^\infty(\mathbf{R}^n). \tag{1.3.5}$$

More generally, if $X \subset \mathbf{R}^n$ is an open set and y is a point of X then $\delta_y \in \mathscr{D}'(X)$ is defined by

$$\langle \delta_y, \phi \rangle = \phi(y), \quad \phi \in C_c^\infty(X). \tag{1.3.6}$$

It is easy to show that neither (1.3.5) nor (1.3.6) can be put into the form (1.3.2); the proof is left to the reader.

There is another way of characterizing distributions. For this, we need a definition.

Definition 1.3.2. Let $X \subset \mathbf{R}^n$ be an open set. A sequence $(\phi_j)_{1 \leqslant j < \infty} \in C_c^\infty(X)$ is said to converge (or tend) to zero in $C_c^\infty(X)$ if (i) the supports of the ϕ_j are contained in a fixed compact subset of X, and (ii) for each multi-index α, the $\partial^\alpha \phi_j$ converge to zero uniformly as $j \to \infty$.

Theorem 1.3.2. A linear form u on $C_c^\infty(X)$ is a distribution if and only if $\lim_{j\to\infty}\langle u, \phi_j\rangle = 0$ for every sequence ϕ_j which converges to zero in $C_c^\infty(X)$ as $j\to\infty$.

Note. This property is called *sequential continuity*.

Proof. It is obvious from Definitions 1.3.1 and 1.3.2 that the condition is necessary. To prove sufficiency, one argues by contradiction. Suppose, then, that u is sequentially continuous, but is not a distribution. Then there is a compact set $K \subset X$ such that, for each nonnegative integer N, the numbers

$$|\langle u, \phi\rangle| / \sum_{|\alpha|\leqslant N} \sup |\partial^\alpha \phi|, \quad \phi \in C_c^\infty(K)$$

are an unbounded subset of $[0, \infty)$. (For the definition of $C_c^\infty(K)$, see (1.3.4).) Hence for each $N = 0, 1, \ldots$, there is a $\phi_N \in C_c^\infty(K)$ such that

$$|\langle u, \phi_N\rangle| \geqslant N \sum_{|\alpha|\leqslant N} \sup |\partial^\alpha \phi_N|. \tag{1.3.7}$$

Put

$$\psi_N(x) = \phi_N(x) / \sum_{|\alpha|\leqslant N} \sup |\partial^\alpha \phi_N|.$$

Then $\operatorname{supp} \psi_N \subset K$, and $|\partial^\beta \psi_N| \leqslant 1/N$ for $|\beta| \leqslant N$. Hence $\psi_N \to 0$ in $C_c^\infty(X)$ as $N\to\infty$. But it follows from (1.3.7) that $|\langle u, \psi_N\rangle| \geqslant 1$ for all N. So we have arrived at a contradiction, and the proof is complete.

At this point, one can introduce a useful subspace of $\mathscr{D}'(X)$, the *distributions of finite order*.

Definition 1.3.3. A distribution $u \in \mathscr{D}'(X)$ is said to be of finite order if one can take the same N in (1.3.1) for all compact sets $K \subset X$; its order is then the least such N. The vector space of distributions of order $\leqslant m$ is called $\mathscr{D}'^m(X)$.

One can also modify Definition 1.3.2 as follows:

Definition 1.3.2′. A sequence $(\phi_j)_{1\leqslant j<\infty} \in C_c^m(X)$ is said to converge to zero in $C_c^m(X)$ if the supports of all the ϕ_j are contained in a fixed compact subset of X, and the $\partial^\alpha \phi_j$, where $|\alpha| \leqslant m$, converge uniformly to zero as $j\to\infty$.

One then has

Theorem 1.3.3. A distribution $u \in \mathscr{D}'^m(X)$, where $0 \leqslant m < \infty$, has a unique extension to a linear form on $C_c^m(X)$ which is sequentially continuous. Conversely, the restriction of a sequentially continuous linear form on $C_c^m(X)$ to $C_c^\infty(X)$ is a member of $\mathscr{D}'^m(X)$.

Proof. If $u \in \mathscr{D}'^m(X)$, then one has semi-norm estimates

$$|\langle u, \phi\rangle| \leqslant C(K) \sum_{|\alpha|\leqslant m} \sup |\partial^\alpha \phi|, \quad \phi \in C_c^\infty(K). \tag{1.3.8}$$

It follows from Theorem 1.2.1 that if $\phi \in C_c^m(X)$ is given, then one can construct a sequence $(\phi_j)_{1\leqslant j<\infty} \in C_c^\infty(X)$ which converges to ϕ in $C_c^m(X)$ as $j\to\infty$. It is clear from (1.3.8) that the numerical sequence $\langle u, \phi_j\rangle$ converges as $j\to\infty$, and that its limit provides a linear form on $C_c^m(X)$ that satisfies (1.3.8) for all $\phi \in C_c^m(X)$; it is also evident that this extension is unique. The second assertion follows from Theorem 1.3.2; the details are left to the reader.

Both our examples, (1.3.2) and (1.3.5), are distributions of order zero. An example of a distribution of order $m>0$ is

$$\phi \mapsto \partial^\alpha\phi(0), \quad \phi\in C_c^\infty(\mathbf{R}^n), \quad |\alpha| = m.$$

Remark. A complex, locally finite Borel measure μ defined on an open set $X\subset \mathbf{R}^n$ determines a distribution by

$$\langle \mu, \phi\rangle = \int \phi \,\mathrm{d}\mu, \quad \phi \in C_c^\infty(X). \tag{1.3.9}$$

Conversely, it follows from Theorem 1.2.1 and the Riesz representation theorem (see [5, p. 39] or [8, p. 119]) that every distribution of order 0 is of the form (1.3.9). The Dirac distribution (1.3.5) is of course a particular instance, and is therefore often called *Dirac measure*.

1.4. Localization

If $X\subset\mathbf{R}^n$ is an open set and X' is an open subset of X, then one has an inclusion $C_c^\infty(X')\to C_c^\infty(X)$, since one can extend any $\phi\in C_c^\infty(X')$ to X by setting $\phi=0$ on $X\backslash \bar{X}'$. So, if $u\in\mathscr{D}'(X)$, then

$$\phi\mapsto\langle u,\phi\rangle, \quad \phi\in C_c^\infty(X')$$

is a distribution on X'; the semi-norm estimates (1.3.1) yield the requisite semi-norm estimates on X'. This is the *restriction* of u to X'. Two distributions u and v defined on X are then said to be *equal on* X' if their restrictions are equal, that is to say if $\langle u,\phi\rangle=\langle v,\phi\rangle$ for all $\phi\in C_c^\infty(X')$.

The restriction of $u\in\mathscr{D}'(X)$ to an open subset of X is also called a *localization*. A distribution can be recovered from its localizations. To prove this, one needs an important technical device, called a partition of unity. The next theorem is a simple version of this, which will be sufficient for our purpose.