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For Maria Antonietta

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Preface

Euclid's *Elements* held sway in the mathematical world for more than two thousand years. (For an English translation, see Euclid (1956).) In the nineteenth century, however, the growing demand for rigour led to a re-examination of the Euclidean edifice and the realization that there were cracks in it. The critical reappraisal which followed was synthesized by Hilbert (1899), in his *Grundlagen der Geometrie*, which has in turn held sway for almost a century.

In what ways may Hilbert's treatment be improved upon today? Apart from technical improvements, a number of which were included in later editions of his book, it may be argued that Hilbert followed Euclid too closely. The number of undefined concepts is unnecessarily large and the axioms of congruence sit uneasily with the other axioms. In addition, the restriction to three-dimensional space conceals the generality of the results and seems artificial now.

For these reasons Hilbert's approach is often replaced today by a purely algebraic one – the axioms for a vector space over an arbitrary field, followed by the axioms for a vector space over the real field with a positive definite scalar product. This permits a rapid development, but it simply begs the question of why it is possible to introduce coordinates on a line. Only when one attempts to answer this question does one realise that many important results in no way depend on the introduction of coordinates. It is curious that some powerful advocates of 'coordinate-free' linear algebra have used real analysis to prove purely geometric results, such as the Hahn–Banach theorem, without any apparent twinge of conscience.

A system of axioms for Euclidean geometry in which the only undefined concepts are *point* and *segment* was already given before Hilbert by Peano. We follow his example in the present work, although some of our axioms differ. The choice of a system of axioms is inherently arbitrary, since there will be many equivalent systems. However, the purpose of an axiom system is not only to provide a basis for rigorous proof, but also to reveal the structure of a subject.

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From this point of view one axiom system may seem preferable to another which is equivalent to it. The development should seem natural, almost inevitable.

Considerations of this nature have led us to isolate a basic structure, here called a *convex geometry*, which is defined by two axioms only. Additional axioms are chosen so that each individually, in conjunction with these, guarantees some important property. Four such additional axioms define another basic structure, here called a *linear geometry*, for which a rich theory may be developed. Examples of linear geometries, in addition to Euclidean space, are hyperbolic space and (hemi)spherical space, where a 'segment' is the geodesic arc joining two points.

Apart from a dimensionality axiom, only seven axioms in all are needed to characterize ordinary Euclidean or, more strictly, *real affine* space. However, dimensionality plays a role in achieving this number, by eliminating some other possibilities in one or two dimensions. Although the characterization of Euclidean space may be regarded as our ultimate goal, it would be contrary to our purpose to impose all the axioms from the outset. Instead we adjoin axioms successively, so that results are proved under minimal hypotheses and may be applied in other situations, which are of interest in their own right. For example, Proposition III.17 establishes the theorems of Helly and Radon in any linear geometry, and Chapter IV similarly extends the facial theory of polytopes. In this way each result appears, so to speak, in its 'proper place'. When there is a branching of paths, we choose the one which leads to our ultimate goal. This approach, I believe, has not previously been pursued in such a systematic manner. I have found it illuminating myself and hope that the illumination succeeds in shining through the present account.

We have spoken of the characterization of Euclidean space as our ultimate goal, but we are actually concerned with characterizing only a convex subset of Euclidean space. The greater freedom of the whole Euclidean space is mathematically desirable, but it is reasonable to require only that our physical world may be embedded in such a space. This type of non-Euclidean geometry was already mentioned by Klein (1873) and was further considered by Schur (1909). In view of the great historical importance of the parallel axiom, it is of interest that it can play no role here, since it fails to hold in any proper convex subset of Euclidean space (except a lower-dimensional Euclidean space).

The foundations of geometry have been studied for thousands of years, and thousands of papers have been written on the subject. Since it is impossible to do

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justice to all previous contributions in such a situation, we have deliberately restricted the number of references. Thus the absence of a reference does not necessarily imply ignorance of its existence and is not a judgement on its quality. Some references are included simply to provide a time scale and others for their useful bibliographies. Our task has been to select and organize, and sometimes extend. Some open problems are mentioned at the ends of Chapters III, IV and VIII.

The topic of axiomatic convexity has been omitted from the recent extremely valuable *Handbook of convex geometry* (Vols. A and B, ed. P.M. Gruber and J.M. Wills, North-Holland, Amsterdam, 1993), and ordered geometry receives only passing reference (on p. 1311) in the equally valuable *Handbook of incidence geometry* (ed. F. Buekenhout, North-Holland, Amsterdam, 1995). It is hoped that the present work may in some measure repair these omissions. Our aim has been to give a connected account of these subjects, which may be read without reference to other sources. Since many results are established under weaker hypotheses than usual, rather detailed proofs have been given in some cases.

The work is not arranged as a textbook, with starred sections and exercises, and is perhaps more difficult than the usual final-year undergraduate or first-year graduate course. However, the mathematical prerequisites are no greater and I believe that an interesting course could be constructed from the material here, which would acquaint students with a cross-section of mathematics in contrast to the usual compartmentalized course.

Familiarity is assumed with the usual language and notation of set theory, with such algebraic concepts as group, field and vector space, and with Dedekind's construction of the real numbers from the rationals. Concepts from other areas, such as partially ordered sets and lattices, projective geometry, topology and metric spaces, are defined in the text. A few elementary properties of metric spaces are stated without proof. On the two or three occasions when appeal is made to some other result the subsequent development is not at stake. Although the work is essentially self-contained, the reader is still encouraged to consult other treatments, both in the references cited in the notes at the end of each chapter and in the introductory chapters of more specialized works on convexity theory, such as Bonnesen and Fenchel (1934), Valentine (1964), Leichtweiss (1980) and Schneider (1993). The remaining prerequisites for the present work are more substantial – the ability and resolve to follow a detailed logical argument and a love of mathematics.

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As I write these lines on the eve of my retirement from paid employment in the Institute of Advanced Studies at the Australian National University, I take the opportunity to acknowledge that this book could not have been written without the privileged working conditions which I have enjoyed.

Andrew Coppel

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