Introduction

In the present work convex subsets of a real vector space are characterized by a small number of axioms involving *points* and *segments*. At the same time a substantial part of the basic theory of convex sets is developed in a purely geometric manner. This contrasts with traditional treatments, in which the origin has a distinguished role and in which use is made of such analytic devices as duality and the difference of two sets, not to mention metric properties.

We owe to Pasch (1882) the recognition of the role of *order* in Euclidean geometry, both the order of points on a line and a more subtle order associated with points in a plane. A basic undefined concept in Pasch's work was the segment determined by two distinct points. By assuming various axioms involving segments he was able to define a *line* and the order of points on a line. An additional undefined concept was that of the *plane* determined by three points not belonging to the same line.

Peano (1889) avoided this additional undefined concept by replacing Pasch's planar ordering axiom by two further axioms involving segments. Thus he was able to give a system of axioms for Euclidean geometry in which the only undefined concepts were point and segment. Some of Peano's axioms were consequences of the remaining axioms. An independent system of axioms, involving only points and segments, was given by Veblen (1904).

Two axioms which were used from the outset by all these authors will be mentioned here. The first, called *density* in the present work, states that the segment $[a,b]$ determined by two distinct points a,b contains a point $c \neq a,b$. The second, called *unendingness* in the present work, states that for any two distinct points a,b there is a point $c \neq a,b$ such that b is contained in the segment $[a,c]$ determined by a,c. In recent years discrete mathematics has attained equal status with continuous mathematics, and it seems desirable to develop the foundations of geometry as far as possible without using these two axioms. This is what is done here.

One consequence is that our segments are closed (endpoints included), rather than open (endpoints excluded) as in the work of the authors cited. However, this is not a novelty; see, for example, Szczerba and Tarski (1979). What is new here is the gradual introduction of axioms and an obstinate insistence on establishing results without the use of axioms which are not required.

I
Alignments

This chapter is of a preparatory nature. For convenience of reference we begin by defining some basic concepts related to partially ordered sets and lattices. We also prove Hausdorff's maximality theorem, which will be used repeatedly. However, the main topic of the chapter is alignments. An *alignment*, like a topology, is a collection of subsets with certain properties. (The name is not very suggestive, but has the merit that it is not used elsewhere in mathematics.) Alignments, or equivalently *algebraic hull operators*, provide an abstract framework for such concepts as *extreme point, independent set, basis* and *face*. *Helly sets, Radon sets* and *Carathéodory sets* are also defined. Although they are of interest in the present general context, they will acquire a more familiar form in Chapter III.

Two particular types of alignment are given special attention, those with the *exchange property* and those with the *anti-exchange property*. Exchange alignments provide an abstract framework for much of vector space theory, including the concepts of *hyperplane* and *dimension*. Exchange alignments on a finite set are well-known under the name of *matroids*. Anti-exchange alignments, which are of more recent vintage, in the same way provide an abstract framework for some aspects of convexity theory. We also study properties of anti-exchange alignments on a finite set, which are here called *antimatroids*.

1 PARTIALLY ORDERED SETS

Although we will make no use of the theory of partially ordered sets and lattices, we will at times use some of the concepts of these subjects and for convenience of reference we state some definitions here.

A set X is said to be *partially ordered* if a binary relation $\leq$ is defined on X with the properties

(O1) $x \leq x$ (reflexivity),
(O2) *if* $x \leq y$ *and* $y \leq x$, *then* $x = y$ (antisymmetry),
(O3) *if* $x \leq y$ *and* $y \leq z$, *then* $x \leq z$ (transitivity).

Here a *binary relation* on X is just a subset R of the product set $X \times X$ and $x \leq y$ denotes that $(x,y) \in R$. It may be that neither $x \leq y$ nor $y \leq x$, in which case x and y are said to be *incomparable*. A partially ordered set is *totally ordered* if no two elements are incomparable.

If $x \leq y$ and $x \neq y$, we write $x < y$. Instead of $x \leq y$ we may write $y \geq x$, and instead of $x < y$ we may write $y > x$.

We say that $x \in X$ is an *upper bound* for a subset Y of a partially ordered set X if $y \leq x$ for every $y \in Y$, and a *lower bound* for Y if $x \leq y$ for every $y \in Y$. An upper bound for Y is said to be a least upper bound, or *supremum*, for Y if it is a lower bound for the set of all upper bounds. Similarly a lower bound for Y is said to be a greatest lower bound, or *infimum*, for Y if it is an upper bound for the set of all lower bounds. It follows from **(O2)** that Y has at most one supremum and at most one infimum.

A partially ordered set is a *lattice* if any two elements x,y have a supremum $x \vee y$ and an infimum $x \wedge y$. It is a *complete lattice* if every subset Y has a supremum and an infimum.

If X and Y are partially ordered sets, a map $f: X \to Y$ is said to be *order-preserving* if $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$. It is said to be an *order isomorphism* if, in addition, it is a bijection and $f(x_1) \leq f(x_2)$ implies $x_1 \leq x_2$. An order isomorphism between two lattices is a *lattice isomorphism*.

The preceding definitions are all rather basic. A few concepts of lesser importance will be defined when they are encountered.

For us the most important example of a partially ordered set is the collection X of all subsets of a given set C, where $A \leq B$ denotes that the subset A is contained in the subset B. This partially ordered set is in fact a complete lattice, since the supremum of any family $\{A_\alpha\}$ of subsets is their union $\bigcup A_\alpha$ and the infimum is their intersection $\bigcap A_\alpha$.

One property of partially ordered sets will be proved here, since it will be used repeatedly. This property may be formulated in several equivalent ways. We choose here a formulation due to Hausdorff (1927) and will always appeal to it. Other formulations, including the well-ordering theorem and the popular Zorn's lemma, may be found in Hewitt and Stromberg (1975).

HAUSDORFF'S MAXIMALITY THEOREM *Every nonempty partially ordered set X contains a maximal totally ordered subset.*

Proof Let $\mathcal{A}$ be the family of all totally ordered subsets of X. Evidently $\mathcal{A}$ is not empty, since it contains any *singleton*, i.e. any subset of X containing exactly one element. If $\mathcal{T}$ is any subfamily of $\mathcal{A}$ which is totally ordered by inclusion, then the union of all the (totally ordered) sets in $\mathcal{T}$ is again totally ordered and hence is in $\mathcal{A}$.

By the axiom of choice there exists a function f which associates to each nonempty subset E of X an element $f(E)$ of E. For any $A \in \mathcal{A}$, let $A^\dagger$ be the set of all $x \in X \backslash A$ such that $A \cup x \in \mathcal{A}$. We put

$$g(A) = \begin{cases} A \cup f(A^\dagger) & \text{if } A^\dagger \neq \emptyset, \\ A & \text{if } A^\dagger = \emptyset. \end{cases}$$

The function $g\colon \mathcal{A} \to \mathcal{A}$ has the property that $A \subseteq g(A)$ and that at most one element of $g(A)$ is not in A. We wish to show that $g(A) = A$ for at least one $A \in \mathcal{A}$, since then $A^\dagger = \emptyset$ and A is a maximal element of $\mathcal{A}$.

Fix $A_0 \in \mathcal{A}$. We will call a subfamily $\mathcal{B}$ of $\mathcal{A}$ a 'tower' if it has the following three properties:

(i) $A_0 \in \mathcal{B}$,
(ii) if $\mathcal{T}$ is any subfamily of $\mathcal{B}$ which is totally ordered by inclusion, then the union of all the sets in $\mathcal{T}$ is again in $\mathcal{B}$,
(iii) if $B \in \mathcal{B}$, then $g(B) \in \mathcal{B}$.

For example, the family of all $B \in \mathcal{A}$ such that $A_0 \subseteq B$ is a tower. If $\mathcal{B}_0$ is the intersection of all towers, then $\mathcal{B}_0$ is itself a tower but no proper subfamily of $\mathcal{B}_0$ is a tower. Evidently $A_0 \subseteq B$ for every $B \in \mathcal{B}_0$. It is enough to show that $\mathcal{B}_0$ is totally ordered by inclusion. For the union $\overline{B}$ of all sets in $\mathcal{B}_0$ will be in $\mathcal{B}_0$, by (ii), and $g(\overline{B}) \in \mathcal{B}_0$, by (iii). Hence $g(\overline{B}) \subseteq \overline{B}$, by the definition of $\overline{B}$. Since $\overline{B} \subseteq g(\overline{B})$, by the definition of g, it follows that $g(\overline{B}) = \overline{B}$.

Let $\mathcal{C}$ be the family of all $C \in \mathcal{B}_0$ such that, for every $B \in \mathcal{B}_0$, either $B \subseteq C$ or $C \subseteq B$. For example, $A_0 \in \mathcal{C}$. For each $C \in \mathcal{C}$, let $\mathcal{D}(C)$ be the family of all $B \in \mathcal{B}_0$ such that either $B \subseteq C$ or $g(C) \subseteq B$. Again, $A_0 \in \mathcal{D}(C)$. Thus (i) is satisfied by $\mathscr{C}$ and by each $\mathscr{D}(C)$. Evidently also (ii) is satisfied by $\mathscr{C}$ and by each $\mathscr{D}(C)$.

To show that $\mathscr{D}(C)$ is a tower it remains to show that $B \in \mathscr{D}(C)$ implies $g(B) \in \mathscr{D}(C)$, i.e. that $B \subseteq C$ and $g(C) \subseteq B$ each imply either $g(B) \subseteq C$ or $g(C) \subseteq g(B)$. If $g(C) \subseteq B$ then also $g(C) \subseteq g(B)$, since $B \subseteq g(B)$. If $B = C$, then $g(C) = g(B)$. If $B \subset C$, then C cannot be a proper subset of $g(B)$, since $g(B) \setminus B$ contains at most one element, and hence $g(B) \subseteq C$, since $C \in \mathscr{C}$ and $g(B) \in \mathscr{B}_0$.

Thus $\mathscr{D}(C)$ is a tower. Since $\mathscr{B}_0$ is minimal, it follows that $\mathscr{D}(C) = \mathscr{B}_0$ for each $C \in \mathscr{C}$. Thus if $B \in \mathscr{B}_0$ and $C \in \mathscr{C}$, then either $B \subseteq C$ or $g(C) \subseteq B$. Thus $g(C) \in \mathscr{C}$ and $\mathscr{C}$ is a tower. Since $\mathscr{B}_0$ is minimal, it follows that $\mathscr{C} = \mathscr{B}_0$. Hence, by the definition of $\mathscr{C}$, $\mathscr{B}_0$ is totally ordered. □

2 ALIGNED SPACES

Let X be a set and let $\mathscr{C}$ be a collection of subsets of X. The collection $\mathscr{C}$ will be said to be an *alignment* on X if it has the following three properties:

(A1) *the set X is itself in $\mathscr{C}$,*
(A2) *the intersection of any nonempty family of sets in $\mathscr{C}$ is again a set in $\mathscr{C}$,*
(A3) *the union of any nonempty family of sets in $\mathscr{C}$ which is totally ordered by inclusion is again a set in $\mathscr{C}$.*

For example, the collection of all subgroups of a group G is an alignment on G, and the collection of all subspaces of a vector space V is an alignment on V.

The subsets of X which are in $\mathscr{C}$ will be said to be *convex*. In subsequent chapters we will study particular types of alignment, and the notion of convex set will be correspondingly restricted.

For any set $S \subseteq X$, let $[S]$ denote the intersection of all convex sets which contain S. Then, by **(A1)–(A2)**, $[S]$ is itself a convex set, which we call the *convex hull* of S.

PROPOSITION 1 *Convex hulls have the following properties*:

(H1) $S \subseteq [S]$,
(H2) $S \subseteq T$ *implies* $[S] \subseteq [T]$,
(H3) $[[S]] = [S]$,

(H4) *the convex hull of any set is the union of the convex hulls of all its finite subsets.*

Proof Properties **(H1)** and **(H2)** follow immediately from the definition of convex hull, and **(H3)** restates that $[S]$ is convex. The proof of **(H4)** requires the use of **(A3)** and is not so immediate.

A set $S \subseteq X$ will be said to be 'good' if $[S] = \bigcup [F]$, where F runs through all finite subsets of S, and will be said to be 'excellent' if $S \cup H$ is good for every finite set $H \subseteq X$. Thus every finite subset of X is excellent, and $S \cup H$ is excellent if S is excellent and H finite.

Let S be an arbitrary nonempty subset of X and let $\mathscr{E}$ be the family of all excellent subsets of S. Then $\mathscr{E}$ is not empty, since it contains every finite subset of S. By Hausdorff's maximality theorem the family $\mathscr{E}$, partially ordered by inclusion, contains a maximal totally ordered subfamily $\mathscr{T}$. Put $M = \bigcup_{T \in \mathscr{T}} T$. We are going to show that $M \in \mathscr{E}$.

If H is any finite subset of X, the family $\{[T \cup H]: T \in \mathscr{T}\}$ is also totally ordered by inclusion. Hence if we put $C_H = \bigcup_{T \in \mathscr{T}} [T \cup H]$, then C_H is convex. Since $M \cup H \subseteq C_H$, it follows that $[M \cup H] \subseteq C_H$. On the other hand, $T \subseteq M$ implies $[T \cup H] \subseteq [M \cup H]$ and hence $C_H \subseteq [M \cup H]$. Thus $C_H = [M \cup H]$. Consequently, if $x \in [M \cup H]$ then $x \in [T \cup H]$ for some $T \in \mathscr{T}$. Since T is excellent, it follows that $x \in [F \cup H]$ for some finite set $F \subseteq T \subseteq M$. Thus $M \cup H$ is good and M is excellent, as we wished to prove.

By the definition of $\mathscr{T}$, no excellent subset of S properly contains M. Since $M \cup s$ is excellent if $s \in S \backslash M$, it follows that $M = S$. Thus S is indeed good. □

The properties **(H1)**–**(H3)** alone imply that, for arbitrary sets $S, T \subseteq X$,

$$[S \cup T] \;=\; [S \cup [T]] \;=\; [[S] \cup [T]].$$

Let 2^X be the collection of all subsets of the set X. A map $h: 2^X \to 2^X$ is said to be a *hull operator* on X if it has the following three properties:

(i) $S \subseteq h(S)$,

(ii) $S \subseteq T$ *implies* $h(S) \subseteq h(T)$,

(iii) $h(h(S)) = h(S)$.

The hull operator is said to be *algebraic* if, in addition,

(iv) *$x \in h(S)$ implies $x \in h(F)$ for some finite set $F \subseteq S$.*

We have shown that from any alignment on X we can derive an algebraic hull operator. However, the process can also be reversed. If $h: 2^X \to 2^X$ is an algebraic hull operator on X, then one sees immediately that the collection $\mathscr{C}$ of all sets $S \subseteq X$ such that $h(S) = S$ is an alignment on X. In fact, there is a bijective correspondence between alignments on X and algebraic hull operators on X. For our purposes it is more convenient to take alignments as the starting-point.

An alignment $\mathscr{C}$ on a set X may be said to be *normed* if it has the additional property

(A0) *the empty set $\varnothing$ is in $\mathscr{C}$.*

There is no loss of generality in restricting attention to normed alignments. For let $\mathscr{C}$ be an arbitrary alignment on X, and put $E = \bigcap_{C \in \mathscr{C}} C$. Then the collection $\mathfrak{D}$ of all sets $\{C \setminus E: C \in \mathscr{C}\}$ is a normed alignment on X, since $E \in \mathscr{C}$.

We will assume throughout the remainder of this section that we are given a set X and a normed alignment $\mathscr{C}$ on X. We will say that the pair $(X, \mathscr{C})$ is an *aligned space*, or simply that X is an aligned space if the meaning is clear. The qualification 'nonempty' in the statement of **(A2)** and **(A3)** may now be omitted, and convex hulls now have not only the properties **(H1)–(H4)** of Proposition 1 but also the property

(H0) $[\varnothing] = \varnothing$.

The collection $\mathscr{C}$ of all convex sets, partially ordered by inclusion, is a *complete lattice*. For any family $\{C_\alpha\}$ of convex sets has an infimum, namely $\bigcap C_\alpha$, and a supremum, namely the intersection of all convex sets which contain $\bigcup C_\alpha$.

If $S \subseteq X$ and $e \in S$, then e is said to be an *extreme point* of the set S if $e \notin [S \setminus e]$. Clearly if e is an extreme point of S, then it is also an extreme point of every subset of S which contains it. From the definition we obtain also

PROPOSITION 2 *Let C be a convex set and $e \in C$. Then e is an extreme point of C if and only if $C \setminus e$ is convex.*

Proof If $C \setminus e$ is convex then $[C \setminus e] = C \setminus e$, and if $C \setminus e$ is not convex then $[C \setminus e] = C$. □

Since any intersection of convex sets is again a convex set, it follows from Proposition 2 that if an arbitrary collection of extreme points is removed from a convex set, then the remaining set is convex.

We will denote by $\mathbf{E}(S)$ the set of all extreme points of the set S. It can be characterized in the following way:

PROPOSITION 3 *If $S \subseteq X$, then $\mathbf{E}(S)$ is the intersection of all subsets of S which have the same convex hull as S.*

Proof If e is an extreme point of S and if T is a subset of S with $[T] = [S]$, then $e \in T$, since $[S \setminus e]$ is a proper subset of $[S]$. On the other hand, if $e \in S$ is not an extreme point of S then $e \in [S \setminus e]$. Hence $S \subseteq [S \setminus e]$ and $[S] = [S \setminus e]$. But $e \notin S \setminus e$. □

A set $S \subseteq X$ will be said to be *independent* if every point of S is an extreme point of S.

It follows at once from this definition that, for an arbitrary set $S \subseteq X$, the set $\mathbf{E}(S)$ of all extreme points of S is an independent set. Furthermore, any singleton is an independent set and any subset of an independent set is again an independent set. On the other hand, it follows from **(H4)** that an infinite set is independent if every finite subset is independent. Consequently the union of a totally ordered collection of independent sets is again an independent set. Hence, by Hausdorff's maximality theorem, any independent subset of a set is contained in a maximal independent subset.

A subset T of a set S will be said to *generate* S if $[T] = [S]$, and will be said to be a *basis* of S if in addition it is independent.

It follows at once from the definitions that a subset T of a set S is a basis of S if and only if T generates S, but no proper subset of T generates S. Hence a set has a finite basis if it is generated by a finite subset. Any basis of a set S is a maximal independent subset of S, but in general the converse is not true.

A subset A of a convex set C will be said to be a *face* of C if A is convex and if, for every set $S \subseteq C$,

$$[S] \cap A \subseteq [S \cap A].$$

It is actually sufficient to require that $[F] \cap A \subseteq [F \cap A]$ for every finite set $F \subseteq C$. For then, if $S \subseteq C$ and $x \in [S] \cap A$, there is a finite set $F \subseteq S$ such that

$$x \in [F] \cap A \subseteq [F \cap A] \subseteq [S \cap A].$$

Obviously C is itself a face. The *proper* faces of a convex set C are the faces other than C itself.

PROPOSITION 4 *The collection of all faces of a convex set C is a normed alignment on C.*

Proof Evidently $\emptyset$ and C are faces of C. If $\{A_i: i \in I\}$ is a family of faces of C and $A = \bigcap_{i \in I} A_i$, then A is a convex subset of C. Suppose $S \subseteq C$ and $x \in [S] \cap A$. Then $x \in [F]$ for some finite $F \subseteq S$. Moreover we may assume that $x \notin [F^*]$ for every proper subset F^* of F. Then, since $x \in [F] \cap A_i \subseteq [F \cap A_i]$, we have $F \cap A_i = F$ for all $i \in I$. Hence $F \subseteq A_i$ for all $i \in I$, and so $F \subseteq A$. Consequently $x \in [F] \subseteq [S \cap A]$, proving that A is a face of C.

If $\{A_i: i \in I\}$ is a family of faces of C which is totally ordered by inclusion and $A = \bigcup_{i \in I} A_i$, then A is a convex subset of C. Suppose $S \subseteq C$ and $x \in [S] \cap A$. Then, for some $i \in I$, $x \in [S] \cap A_i \subseteq [S \cap A_i]$. Hence $[S] \cap A \subseteq [S \cap A]$, and A is a face of C. □

The collection of all faces of a convex set C, partially ordered by inclusion, is also a complete lattice, since any family $\{F_\alpha\}$ of faces has both an infimum, namely $\bigcap F_\alpha$, and a supremum, namely the intersection of all faces which contain $\bigcup F_\alpha$.

PROPOSITION 5 *If A is a face of the convex set C, then $C \setminus A$ is convex.*

Proof Taking $S = C \setminus A$ in the definition of a face, we obtain $[C \setminus A] \cap A = \emptyset$. Since $[C \setminus A] \subseteq C$, it follows that $[C \setminus A] = C \setminus A$. □

PROPOSITION 6 *If A,B,C are convex sets such that A is a face of B and B is a face of C, then A is a face of C.*

Proof Suppose $S \subseteq C$ and $x \in [S] \cap A$. Then $x \in [S] \cap B$ and hence $x \in [S \cap B]$, since B is a face of C. From $S \cap B \subseteq B$ and $x \in [S \cap B] \cap A$ it follows that $x \in [S \cap A]$, since A is a face of B. □

PROPOSITION 7 *If A is a face of the convex set C and B a face of the convex set D, then $A \cap B$ is a face of the convex set $C \cap D$.*

In particular, if A is a face of the convex set C, then A is a face of any convex set B such that $A \subseteq B \subseteq C$.