The Theory of Partitions

The Theory of Partitions

George E. Andrews

Pennsylvania State University
University Park, Pennsylvania

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge CB2 1RP

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, United Kingdom
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

Original hardcover edition first published 1976 by Addison-Wesley Publishing Company, Inc.
First paperback edition published 1998 by Cambridge University Press

Library of Congress Cataloging-in-Publication Data is available.

A catalog record for this book is available from the British Library.

ISBN 0-521-63766-X paperback

Transferred to digital printing 2003

To Joy

Amy, and

Katy

Editor's Statement

A large body of mathematics consists of facts that can be presented and described much like any other natural phenomenon. These facts, at times explicitly brought out as theorems, at other times concealed within a proof, make up most of the applications of mathematics, and are the most likely to survive changes of style and of interest.

This ENCYCLOPEDIA will attempt to present the factual body of all mathematics. Clarity of exposition, accessibility to the non-specialist, and a thorough bibliography are required of each author. Volumes will appear in no particular order, but will be organized into sections, each one comprising a recognizable branch of present-day mathematics. Numbers of volumes and sections will be reconsidered as times and needs change.

It is hoped that this enterprise will make mathematics more widely used where it is needed, and more accessible in fields in which it can be applied but where it has not yet penetrated because of insufficient information.

The theory of partitions is one of the very few branches of mathematics that can be appreciated by anyone who is endowed with little more than a lively interest in the subject. Its applications are found wherever discrete objects are to be counted or classified, whether in the molecular and the atomic studies of matter, in the theory of numbers, or in combinatorial problems from all sources.

Professor Andrews has written the first thorough survey of this many-sided field. The specialist will consult it for the more recondite results, the student will be challenged by many a deceptively simple fact, and the applied scientist may locate in it the missing identity he needs to organize his data.

Professor Turán's untimely death has left this book without a suitable introduction. It is fitting to dedicate it to the memory of one of the masters of number theory.

GIAN-CARLO ROTA

Contents

Preface to the Paperback Edition

In the past twenty years, the theory of partitions has blossomed. The object of this book, as appropriate in this series, is to provide the fundamentals in a form accessible to the nonspecialist, with references to the recent literature for those who wish to pursue a particular interest. The major changes I would have made in a total revision would have added greatly to the length of the book. This is a task of such magnitude that it will have to wait until a number of my other projects are completed.

In the light of this introduction, here are my comments on the chapters of the third printing of *The Theory of Partitions*. Chapter 1 contains the almost immutable basics. Chapter 2 is partially devoted to basic hypergeometric series. As such, it is a small introduction to the wonderful world of q that has been so beautifully chronicled by Gasper and Rahman (1990). Andrews (1986) provides a further survey of the interactions of partitions with q. For Chapter 3, it should be pointed out that O'Hara (1990) has shown how to prove the unimodality of Gaussian polynomials in a purely elementary (although not easy) manner. Chapters 3 through 5 are fairly current introductions to their topics.

The work in Chapter 6 has been greatly extended by Richard McIntosh in a series of papers: cf. McIntosh (1995). The material in Chapters 7, 8, and 9 has also been greatly extended in the past decade. Workers in partitions proper (for example, Alladi and Gordon) have made major advances. In the 1970s and 1980s, David Bressoud made spectacular progress in this area. For further references and an account of his central methods, see Bressoud (1980). In addition, the world of statistical mechanics has provided a flood of results, as well as important questions on this topic. The fountainhead of this work is Rodney Baxter (1982).

The introduction to Chapter 10 is reasonably up to date. However, there have been major recent discoveries for partition function congruences; the most notable have been found by Garavan, Kim, and Stanton (1990); Ono (1996); and Gordon and Ono (1997).

There have also been extensive discoveries for higher-dimensional partitions. Here the interested reader should consult surveys by Stanley (1986a, 1986b) and a related paper by Robbins (1991). As many of the conjectures in these papers are now theorums, these surveys are themselves becoming out of date, but they are good leads for what has been done.

The topics in Chapters 12 and 13 might well be augmented with an account of generalized Frobenius partitions (see Andrews [1984]).

Chapter 14 contains the appropriate computational rudiments. However, Zeilberger (1991) has pioneered exciting new computational work related to partitions and other combinatorial identities (cf. Zeilberger and Wilf [1990]).

The following list of references is designed only to provide the reader with leads into the literature. It does not in any way give adequate credit to the numerous contributions of scores of researchers during the past two decades.

References

Alladi, K. (1995). "The method of weighted words and applications to partitions," *Number Theory* (S. David, ed.). Cambridge University Press, Cambridge.

Andrews, G. E. (1984). "Generalized Frobenius partitions," *Mem. Amer. Math. Soc.* **301.**

Andrews, G. E. (1986). "q-Series: Their development and application in analysis, number theory, combinatorics, physics and computer algebra," *Mem. Amer. Math. Soc.* **66.**

Baxter, R. J. (1982). *Exactly Solved Models in Statistical Mechanics*. Academic Press, London and New York.

Bressoud, D. M. (1980a). "Analytic and combinatorial generalizations of the Rogers-Ramanujan identities," *Mem. Amer. Math. Soc.* **227**.

Bressoud, D. M. (1980b). "Extension of the partition sieve," *J. Number Th.* **12,** 76–100.

Bressoud, D. M. (1998). *Proofs and Confirmations*. Cambridge University Press, Cambridge.

Burge, W. H. (1981). "A correspondence between partitions related to generalizations of the Rogers-Ramanujan identities," *Discrete Math.* **34,** 9–15.

Garvan, F., Kim, D., and Stanton, D. (1990). "Cranks and t-cores," *Invent. Math.* **101,** 1–17.

Gasper, G, and Rahman, M. (1990). "Basic hypergeometric series," *Encycl. Math and Its Applications* **35** (G-C. Rota, ed.). Cambridge University Press, Cambridge.

Gordon, B., and Ono, K. (1997). "Divisibility properties of certain partition functions by powers of primes," *Ramunujan J.* **1,** 25–35.

McIntosh, R. J. (1995). "Some asymptotic formulae for q-hypergeometric series," *J. London Math. Soc. (2)* **51,** 120–136.

O'Hara, K. M. (1990). "Unimodality of Gaussian coefficients: A constructive proof," *J. Combinatorial Th.* **A53,** 29–52.

Ono, K. (1996). "On the parity of the partition function in arithmetic progressions," *J. Reine Angew. Math.* **472,** 1–15.

Robbins, D. P. (1991). "The story of 1, 2, 7, 42, 429, 7436, . . .," *Math. Intellig.* **13,** 12–19.

Stanley, R. P. (1986a). "A baker's dozen of conjectures concerning plane partitions," *Lecture Notes in Math.* **1234,** 285–293. Springer, Berlin.

Stanley, R. P. (1986b) "Symmetries of plane partitions," *J. Comb. Th.* **A43,** 103–113.

Zeilberger, D., and Wilf, H. S. (1990). "Rational functions certify combinatorial identities," *J. Amer. Math. Soc.* **3,** 147–158.

Zeilberger, D. (1991). "The method of creative telescoping," *J. Sym. Comp.* **11,** 195–204.

Preface

Let us begin by acknowledging that the word "partition" has numerous meanings in mathematics. Any time a division of some object into subobjects is undertaken, the word partition is likely to pop up. For the purposes of this book a "partition of n" is a nonincreasing finite sequence of positive integers whose sum is n. We shall extend this definition in Chapters 11, 12, and 13 when we consider higher-dimensional partitions, partitions of n-tuples, and partitions of sets, respectively. Compositions or ordered partitions (merely finite sequences of positive integers) will be considered in Chapter 4.

The theory of partitions has an interesting history. Certain special problems in partitions certainly date back to the Middle Ages; however, the first discoveries of any depth were made in the eighteenth century when L. Euler proved many beautiful and significant partition theorems. Euler indeed laid the foundations of the theory of partitions. Many of the other great mathematicians—Cayley, Gauss, Hardy, Jacobi, Lagrange, Legendre, Littlewood, Rademacher, Ramanujan, Schur, and Sylvester—have contributed to the development of the theory.

There have been almost no books devoted entirely to partitions. Generally the combinatorial and formal power series aspects of partitions have found a place in older books on elementary analysis (*Introductio in Analysin Infinitorum* by Euler, *Textbook of Algebra* by Chrystal), in encyclopedic surveys of number theory (*Niedere Zahlentheorie* by Bachman, *Introduction to the Theory of Numbers* by Hardy and Wright), and in combinatorial analysis books (*Combinatory Analysis* by MacMahon, *Introduction to Combinatorial Analysis* by Riordan, *Combinatorial Methods* by Percus, *Advanced Combinatorics* by Comtet). The asymptotic problems associated with partitions have, on the other hand, been treated in works on analytic or additive number theory (*Introduction to the Analytic Theory of Numbers* by Ayoub, *Modular Functions in Analytic Number Theory* by Knopp, *Topics from the Theory of Numbers* by Grosswald, *Additive Zahlentheorie* by Ostmann, *Topics in Analytic Number Theory* by Rademacher).

If one considers the applications of partitions in various branches of mathematics and statistics, one is struck by the interplay of combinatorial and asymptotic methods. We have tried to organize this book so that it adequately develops and interrelates both combinatorial and analytic methods.

Chapters 1–4 treat the elementary portions of the theory of partitions; of primary importance here is the use of generating functions.

Chapters 5 and 6 treat the asymptotic problems. Partition identities are dealt with in Chapters 7 through 9. Chapter 10 on partition function congruences returns to the analytic aspect of partitions. Chapters 11–13 treat several generalizations of partitions and Chapter 14 presents a brief discussion of the computational aspect of partitions.

There are three concluding sections of each chapter: A "Notes" section provides historical comment on the material covered; a "References" section provides a substantial but nonexhaustive list of relevant books and papers; and an "Examples" section provides statements of results not fully covered in the text. Those examples that occur with an asterisk are significant advances beyond the material presented in the text; the remainder form a reasonable set of exercises by which the reader may determine his grasp of the subject matter. References for the source of the examples occur in the related Notes section.

Many of the mathematical sciences have seen applications of partitions recently. Nonparametric statistics require restricted partitions like those in Chapter 3. Various permutation problems in probability and statistics are intimately linked with the Simon Newcomb problem of Chapter 4. Particle physics uses partition asymptotics and partition identities related to the work in Chapters 5–9. Group theory (through Young tableaux) is intimately connected with Chapter 12, and the relationship between partitions and combinatorial theory is explored in Chapter 13.

The material in this book has been developed over a period of years. My first acquaintance with partitions came from thrilling lectures delivered by my thesis adviser, the late Professor Hans Rademacher. Many of the topics herein have been presented in graduate courses at the Pennsylvania State University between 1964 and 1975, in seminars at MIT during the 1970–1971 academic year, at the University of Erlangen in the summer of 1975, and at the University of Wisconsin during the 1975–1976 academic year. I owe a great debt of gratitude to many people at these four universities. I wish to thank specially R. Askey, K. Baclawski, B. Berndt, and L. Carlitz, who contributed many valuable suggestions and comments during the preparation of this book.

Finally I thank my wife, Joy, who has throughout this project been both a help and an inspiration to me.

GEORGE E. ANDREWS