CHAPTER 1

The Elementary Theory of Partitions

1.1 Introduction

In this book we shall study in depth the fundamental additive decomposition process: the representation of positive integers by sums of other positive integers.

DEFINITION 1.1. A *partition* of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the *parts* of the partition.

Many times the partition $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ will be denoted by λ, and we shall write $\lambda \vdash n$ to denote "λ is a partition of n." Sometimes it is useful to use a notation that makes explicit the number of times that a particular integer occurs as a part. Thus if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$, we sometimes write

$$\lambda = (1^{f_1} 2^{f_2} 3^{f_3} \cdots)$$

where exactly f_i of the λ_j are equal to i. Note now that $\sum_{i \geq 1} f_i i = n$.

Numerous types of partition problems will concern us in this book; however, among the most important and fundamental is the question of enumerating various sets of partitions.

DEFINITION 1.2. The partition function $p(n)$ is the number of partitions of n.

Remark. Obviously $p(n) = 0$ when n is negative. We shall set $p(0) = 1$ with the observation that the empty sequence forms the only partition of zero. The following list presents the next six values of $p(n)$ and tabulates the actual partitions.

$p(1) = 1$: $1 = (1)$;
$p(2) = 2$: $2 = (2)$, $1 + 1 = (1^2)$;
$p(3) = 3$: $3 = (3)$, $2 + 1 = (12)$, $1 + 1 + 1 = (1^3)$;
$p(4) = 5$: $4 = (4)$, $3 + 1 = (13)$, $2 + 2 = (2^2)$,
$2 + 1 + 1 = (1^2 2)$, $1 + 1 + 1 + 1 = (1^4)$;

ENCYCLOPEDIA OF MATHEMATICS and Its Applications, Gian-Carlo Rota (ed.). 2, George E. Andrews, The Theory of Partitions

$$
\begin{aligned}
p(5) = 7:\quad & 5 = (5),\quad 4+1 = (14),\quad 3+2 = (23),\\
& 3+1+1 = (1^3 3),\quad 2+2+1 = (12^2),\\
& 2+1+1+1 = (1^3 2),\quad 1+1+1+1+1 = (1^5);\\
p(6) = 11:\quad & 6 = (6),\quad 5+1 = (15),\quad 4+2 = (24),\\
& 4+1+1 = (1^2 4),\quad 3+3 = (3^2),\quad 3+2+1 = (123),\\
& 3+1+1+1 = (1^3 3),\quad 2+2+2 = (2^3),\\
& 2+2+1+1 = (1^2 2^2),\quad 2+1+1+1+1 = (1^4 2),\\
& 1+1+1+1+1+1 = (1^6).
\end{aligned}
$$

The partition function increases quite rapidly with n. For example, $p(10) = 42$, $p(20) = 627$, $p(50) = 204226$, $p(100) = 190569292$, and $p(200) = 3972999029388$.

Many times we are interested in problems in which our concern does not extend to all partitions of n but only to a particular subset of the partitions of n.

DEFINITION 1.3. Let $\mathscr{S}$ denote the set of all partitions.

DEFINITION 1.4. Let $p(S, n)$ denote the number of partitions of n that belong to a subset S of the set $\mathscr{S}$ of all partitions.

For example, we might consider $\mathscr{O}$ the set of all partitions with odd parts and $\mathscr{D}$ the set of all partitions with distinct parts. Below we tabulate partitions related to $\mathscr{O}$ and to $\mathscr{D}$.

$$
\begin{aligned}
p(\mathscr{O}, 1) = 1:\quad & 1 = (1),\\
p(\mathscr{O}, 2) = 1:\quad & 1+1 = (1^2),\\
p(\mathscr{O}, 3) = 2:\quad & 3 = (3),\quad 1+1+1 = (1^3),\\
p(\mathscr{O}, 4) = 2:\quad & 3+1 = (13),\quad 1+1+1+1 = (1^4),\\
p(\mathscr{O}, 5) = 3:\quad & 5 = (5),\quad 3+1+1 = (1^2 3),\\
& 1+1+1+1+1 = (1^5),\\
p(\mathscr{O}, 6) = 4:\quad & 5+1 = (15),\quad 3+3 = (3^2),\\
& 3+1+1+1 = (1^3 3),\\
& 1+1+1+1+1+1 = (1^6),\\
p(\mathscr{O}, 7) = 5:\quad & 7 = (7),\quad 5+1+1 = (1^2 5),\quad 3+3+1 = (13^2),\\
& 3+1+1+1+1 = (1^4 3),\\
& 1+1+1+1+1+1+1 = (1^7).
\end{aligned}
$$

$$
\begin{aligned}
p(\mathscr{D}, 1) = 1:\quad & 1 = (1),\\
p(\mathscr{D}, 2) = 1:\quad & 2 = (2),\\
p(\mathscr{D}, 3) = 2:\quad & 3 = (3),\quad 2+1 = (12),\\
p(\mathscr{D}, 4) = 2:\quad & 4 = (4),\quad 3+1 = (13),\\
p(\mathscr{D}, 5) = 3:\quad & 5 = (5),\quad 4+1 = (14),\quad 3+2 = (23),\\
p(\mathscr{D}, 6) = 4:\quad & 6 = (6),\quad 5+1 = (15),\quad 4+2 = (24),\\
& 3+2+1 = (123),\\
p(\mathscr{D}, 7) = 5:\quad & 7 = (7),\quad 6+1 = (16),\quad 5+2 = (25),\\
& 4+3 = (34),\quad 4+2+1 = (124).
\end{aligned}
$$

We point out the rather curious fact that $p(\mathcal{O}, n) = p(\mathcal{D}, n)$ for $n \leqslant 7$, although there is little apparent relationship between the various partitions listed (see Corollary 1.2).

In this chapter, we shall present two of the most elemental tools for treating partitions: (1) infinite product generating functions; (2) graphical representation of partitions.

1.2 Infinite Product Generating Functions of One Variable

DEFINITION 1.5. The generating function $f(q)$ for the sequence $a_0, a_1, a_2, a_3, \ldots$ is the power series $f(q) = \sum_{n \geqslant 0} a_n q^n$.

Remark. For many of the problems we shall encounter, it suffices to consider $f(q)$ as a "formal power series" in q. With such an approach many of the manipulations of series and products in what follows may be justified almost trivially. On the other hand, much asymptotic work (see Chapter 6) requires that the generating functions be analytic functions of the complex variable q. In actual fact, both approaches have their special merits (recently, E. Bender (1974) has discussed the circumstances in which we may pass from one to the other). Generally we shall state our theorems on generating functions with explicit convergence conditions. For the most part we shall be dealing with absolutely convergent infinite series and infinite products; consequently, various rearrangements of series and interchanges of summation will be justified analytically from this simple fact.

DEFINITION 1.6. Let H be a set of positive integers. We let "H" denote the set of all partitions whose parts lie in H. Consequently, $p(``H", n)$ is the number of partitions of n that have all their parts in H.

Thus if H_0 is the set of all odd positive integers, then "H_0" $= \mathcal{O}$.

$$p(``H_0", n) = p(\mathcal{O}, n).$$

DEFINITION 1.7. Let H be a set of positive integers. We let "H"$(\leqslant d)$ denote the set of all partitions in which no part appears more than d times and each part is in H.

Thus if N is the set of all positive integers, then $p(``N"(\leqslant 1), n) = p(\mathcal{D}, n)$.

THEOREM 1.1. *Let H be a set of positive integers, and let*

$$f(q) = \sum_{n \geqslant 0} p(``H", n) q^n, \tag{1.2.1}$$

$$f_d(q) = \sum_{n \geqslant 0} p(``H"(\leqslant d), n) q^n. \tag{1.2.2}$$

Then for $|q| < 1$

$$f(q) = \prod_{n \in H} (1 - q^n)^{-1}, \tag{1.2.3}$$

$$f_d(q) = \prod_{n \in H} (1 + q^n + \cdots + q^{dn})$$

$$= \prod_{n \in H} (1 - q^{(d+1)n})(1 - q^n)^{-1}. \tag{1.2.4}$$

Remark. The equivalence of the two forms for $f_d(q)$ follows from the simple formula for the sum of a finite geometric series:

$$1 + x + x^2 + \cdots + x^r = \frac{1 - x^{r+1}}{1 - x}.$$

Proof. We shall proceed in a formal manner to prove (1.2.3) and (1.2.4); at the conclusion of our proof we shall sketch how to justify our steps analytically. Let us index the elements of H, so that $H = \{h_1, h_2, h_3, h_4, \ldots\}$. Then

$$\prod_{n \in H} (1 - q^n)^{-1} = \prod_{n \in H} (1 + q^n + q^{2n} + q^{3n} + \cdots)$$

$$= (1 + q^{h_1} + q^{2h_1} + q^{3h_1} + \cdots)$$

$$\times (1 + q^{h_2} + q^{2h_2} + q^{3h_2} + \cdots)$$

$$\times (1 + q^{h_3} + q^{2h_3} + q^{3h_3} + \cdots)$$

$$\cdots$$

$$= \sum_{a_1 \geq 0} \sum_{a_2 \geq 0} \sum_{a_3 \geq 0} \cdots q^{a_1 h_1 + a_2 h_2 + a_3 h_3 + \cdots}$$

and we observe that the exponent of q is just the partition $(h_1{}^{a_1} h_2{}^{a_2} h_3{}^{a_3} \cdots)$. Hence q^N will occur in the foregoing summation once for each partition of n into parts taken from H. Therefore

$$\prod_{n \in H} (1 - q^n)^{-1} = \sum_{n \geq 0} p(\text{“}H\text{”}, n) q^n.$$

The proof of (1.2.4) is identical with that of (1.2.3) except that the infinite geometric series is replaced by the finite geometric series:

$$\prod_{n \in H} (1 + q^n + q^{2n} + \cdots + q^{dn})$$

$$= \sum_{d \geq a_1 \geq 0} \sum_{d \geq a_2 \geq 0} \sum_{d \geq a_3 \geq 0} \cdots q^{a_1 h_1 + a_2 h_2 + a_3 h_3 + \cdots}$$

$$= \sum_{n \geq 0} p(\text{“}H\text{”}(\leqslant d), n) q^n.$$

If we are to view the foregoing procedures as operations with convergent infinite products, then the multiplication of infinitely many series together requires some justification. The simplest procedure is to truncate the infinite product to $\prod_{i=1}^{n}(1-q^{h_i})^{-1}$. This truncated product will generate those partitions whose parts are among $h_1, h_2, \ldots, h_n$. The multiplication is now perfectly valid since only a finite number of absolutely convergent series are involved. Now assume q is real and $0 < q < 1$; then if $M = h_n$,

$$\sum_{j=0}^{M} p(\text{“}H\text{”}, j)q^j \leqslant \prod_{i=1}^{n}(1-q^{h_i})^{-1} \leqslant \prod_{i=1}^{\infty}(1-q^{h_i})^{-1} < \infty.$$

Thus the sequence of partial sums $\sum_{j=0}^{M} p(\text{“}H\text{”}, j)q^j$ is a bounded increasing sequence and must therefore converge. On the other hand

$$\sum_{j=0}^{\infty} p(\text{“}H\text{”}, j)q^j \geqslant \prod_{i=1}^{n}(1-q^{h_i})^{-1} \to \prod_{i=1}^{\infty}(1-q^{h_i})^{-1} \quad \text{as} \quad n \to \infty.$$

Therefore

$$\sum_{j=0}^{\infty} p(\text{“}H\text{”}, j)q^j = \prod_{i=1}^{\infty}(1-q^{h_i})^{-1} = \prod_{n \in H}(1-q^n)^{-1}.$$

Similar justification can be given for the proof of (1.2.4). ■

COROLLARY 1.2 (Euler). *$p(\mathscr{O}, n) = p(\mathscr{D}, n)$ for all n.*

Proof. By Theorem 1.1,

$$\sum_{n \geqslant 0} p(\mathscr{O}, n)q^n = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}$$

and

$$\sum_{n \geqslant 0} p(\mathscr{D}, n)q^n = \prod_{n=1}^{\infty}(1+q^n).$$

Now

$$\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^n)} = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}. \tag{1.2.5}$$

Hence

$$\sum_{n \geqslant 0} p(\mathscr{O}, n)q^n = \sum_{n \geqslant 0} p(\mathscr{D}, n)q^n,$$

and since a power series expansion of a function is unique, we see that $p(\mathscr{C}, n) = p(\mathscr{D}, n)$ for all n. ■

COROLLARY 1.3 (Glaisher). *Let N_d denote the set of those positive integers not divisible by d. Then*

$$p(\text{“}N_{d+1}\text{”}, n) = p(\text{“}N\text{”}(\leqslant d), n)$$

for all n.

Proof. By Theorem 1.1,

$$\begin{aligned}\sum_{n\geqslant 0} p(\text{“}N\text{”}(\leqslant d), n)q^n &= \prod_{n=1}^{\infty} \frac{(1-q^{(d+1)n})}{(1-q^n)}\\ &= \prod_{\substack{n=1\\(d+1)\nmid n}}^{\infty} \frac{1}{(1-q^n)}\\ &= \sum_{n\geqslant 0} p(\text{“}N_{d+1}\text{”}, n)q^n,\end{aligned}$$

and the result follows as before. ■

There are numerous results of the type typified by Corollaries 1.2 and 1.3. We shall run into such results again in Chapters 7 and 8, where much deeper theorems of a similar nature will be discussed.

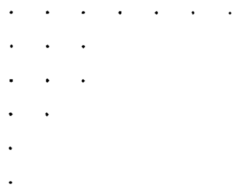
1.3 Graphical Representation of Partitions

Another effective elementary device for studying partitions is the graphical representation. To each partition λ is associated its *graphical representation* $\mathscr{G}_\lambda$ (or Ferrers graph), which formally is the set of points with integral coordinates (i, j) in the plane such that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, then $(i, j) \in \mathscr{G}_\lambda$ if and only if $0 \geqslant i \geqslant -n+1$, $0 \leqslant j \leqslant \lambda_{|i|+1} - 1$. Rather than dwell on this formal definition, we shall, by means of a few examples, fully explain the graphical representation.

The graphical representation of the partition $8 + 6 + 6 + 5 + 1$ is

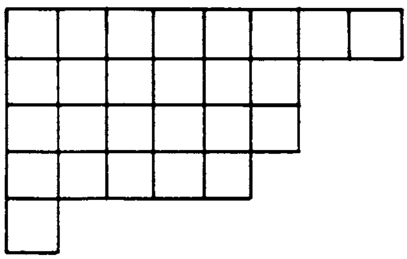
$$\begin{array}{llllllll} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & & & & & & & \end{array}$$

The graphical representation of the partition $7 + 3 + 3 + 2 + 1 + 1$ is



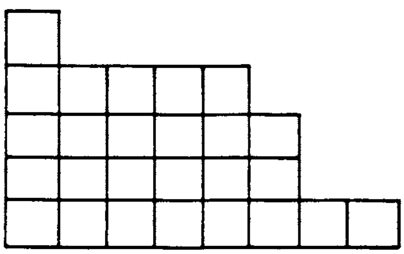
Note that the ith row of the graphical representation of $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ contains λ_i points (or dots, or nodes).

We remark that there are several equivalent ways of forming the graphical representation. Some authors use unit squares instead of points, so that the graphical representation of $8 + 6 + 6 + 5 + 1$ becomes

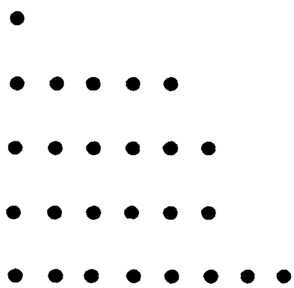


Such a representation is extremely useful when we consider applications of partitions to plane partitions or Young tableaux (see Chapter 11).

Other authors prefer the representation to be upside down (they would say right side up); for example, in the case of $8 + 6 + 6 + 5 + 1$



or



Since most of the classical texts on partitions use the first representation shown in this section, we shall also.

DEFINITION 1.8. If $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition, we may define a new partition $\lambda' = (\lambda_1', \ldots, \lambda_m')$ by choosing λ_i' as the number of parts of λ that are $\geqslant i$. The partition λ' is called the *conjugate* of λ.

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While the formal definition of conjugate is not too revealing, we may better understand the conjugate by using graphical representation. From the definition, we see that the conjugate of the partition $8 + 6 + 6 + 5 + 1$ is $5 + 4 + 4 + 4 + 4 + 3 + 1 + 1$. The graphical representation of $8 + 6 + 6 + 5 + 1$ is

$$\begin{array}{llllllll}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & \\
\cdot & & & & & & &
\end{array}$$

and the conjugate of this partition is obtained by counting the dots in successive columns; that is, the graphical representation of the conjugate is obtained by reflecting the graph in the main diagonal. Thus the graph of the conjugate partition is

$$\begin{array}{lllll}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & & \\
\cdot & & & & \\
\cdot & & & &
\end{array}$$

Notice that not only does the graphical representation provide a simple method by which to obtain the conjugate of λ, but it also shows directly that the conjugate partition λ' is a partition of the same integer as λ is; that is, $\Sigma\lambda_i = \Sigma\lambda_i'$. Furthermore, it is clear that conjugation is an involution of the partitions of any integer, in that the conjugate of the conjugate of λ is again λ.

Let us now prove some theorems on partitions, using graphical representation.

THEOREM 1.4. *The number of partitions of n with at most m parts equals the number of partitions of n in which no part exceeds m.*

Proof. We may set up a one-to-one correspondence between the two classes of partitions under consideration by merely mapping each partition onto its conjugate. The mapping is certainly one-to-one, and by considering the graphical representation we see that under conjugation the condition "at most m parts" is transformed into "no part exceeds m" and vice versa. ■

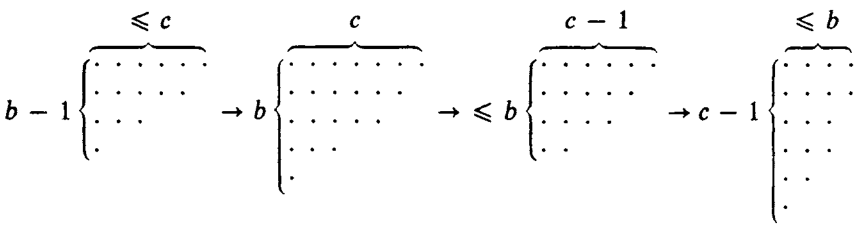
As an example, let us consider the partitions of 6, first into at most three parts and then into parts none of which exceeds 3. We shall list conjugates opposite each other.

6	1 + 1 + 1 + 1 + 1 + 1
5 + 1	2 + 1 + 1 + 1 + 1
4 + 2	2 + 2 + 1 + 1
4 + 1 + 1	3 + 1 + 1 + 1
3 + 3	2 + 2 + 2
3 + 2 + 1	3 + 2 + 1
2 + 2 + 2	3 + 3

Theorem 1.4 is quite useful and shows how a graphical representation can be used directly to obtain important information. More subtle uses of this technique can be seen in the following two theorems.

THEOREM 1.5. *The number of partitions of* $a - c$ *into exactly* $b - 1$ *parts, none exceeding* c, *equals the number of partitions of* $a - b$ *into* $c - 1$ *parts, none exceeding* b.

Proof. Let us consider the graphical representation of a typical partition of the first type mentioned in the theorem. We transform the partition as follows: first we adjoin a new top row of c nodes; then we delete the first column (which now has b nodes); and then we take the conjugate:



We see immediately that this composite transformation provides a one-to-one correspondence between the two types of partitions considered, and consequently the theorem is established. ∎

As an example, let us consider the case in which $a = 14$, $b = 5$, $c = 4$.

4 + 4 + 1 + 1	→	3 + 3 + 3
4 + 3 + 2 + 1	→	4 + 3 + 2
4 + 2 + 2 + 2	→	5 + 2 + 2
3 + 3 + 3 + 1	→	4 + 4 + 1
3 + 3 + 2 + 2	→	5 + 3 + 1

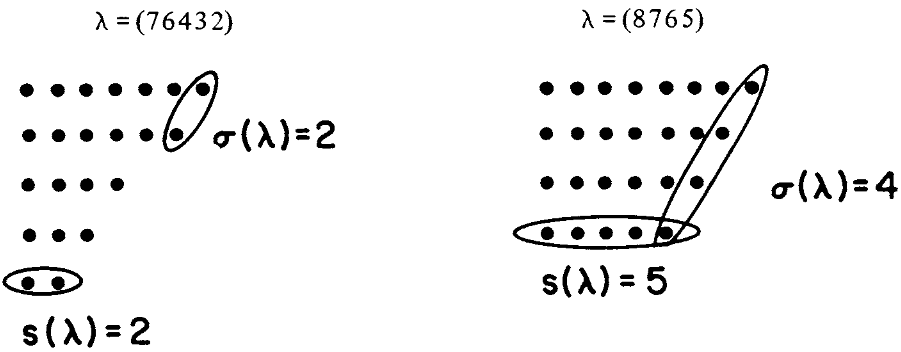
We conclude this chapter with one of the truly remarkable achievements of nineteenth-century American mathematics: F. Franklin's proof of Euler's pentagonal number theorem. Franklin's accomplishment was to prove Legendre's combinatorial interpretation of Euler's theorem. We shall actually state the pentagonal number theorem as Corollary 1.7, and we shall show how useful the theorem is computationally in Corollary 1.8.

THEOREM 1.6. *Let $p_e(\mathscr{D}, n)$ (resp. $p_o(\mathscr{D}, n)$) denote the number of partitions of n into an even (resp. odd) number of distinct parts. Then*

$$p_e(\mathscr{D}, n) - p_o(\mathscr{D}, n) = \begin{cases} (-1)^m & \text{if} \quad n = \frac{1}{2}m(3m \pm 1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We shall attempt to establish a one-to-one correspondence between the partitions enumerated by $p_e(\mathscr{D}, n)$ and those enumerated by $p_o(\mathscr{D}, n)$. For most integers n our attempt will be successful; however, whenever n is one of the pentagonal numbers $\frac{1}{2}m(3m \pm 1)$, a single exceptional case will arise.

To begin with, we note that each partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of n has a smallest part $s(\lambda) = \lambda_r$; also, we observe that the largest part λ_1 of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is the first of a sequence of, say, $\sigma(\lambda)$ consecutive integers that are parts of λ (formally $\sigma(\lambda)$ is the largest j such that $\lambda_j = \lambda_1 - j + 1$). Graphically the parameters $s(\lambda)$ and $\sigma(\lambda)$ are easily described:



We transform partitions as follows.

Case 1. $s(\lambda) \leqslant \sigma(\lambda)$. In this event, we add one to each of the $s(\lambda)$ largest parts of λ and we delete the smallest part. Thus

$$\lambda = (76432) \to \lambda' = (8743);$$

that is