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Representations and Cohomology

II. Cohomology of groups and modules

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CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge CB2 1RP, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, United Kingdom
40 West 20th Street, New York, NY 10011–4211, USA
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1991
First paperback edition 1998

A catalogue record for this book is available from the British Library

ISBN 0 521 36135 4 hardback
ISBN 0 521 63652 3 paperback

Transferred to digital printing 2004

Contents

Introduction

This is the second of two volumes which have grown out of about seven years of graduate courses on various aspects of representation theory and cohomology of groups, given at Yale, Northwestern and Oxford. In this second volume, we concentrate on cohomology of groups and modules. We try to develop everything from both an algebraic and a topological viewpoint, and demonstrate the connection between the two approaches. Having in mind the die-hard algebraist who refuses to have anything to do with topology, we have tried to make sure that if the reader omits all sections involving topology, the rest is still a coherent treatment of the subject. But by trying to present the topology with as few prerequisites as possible, we hope to entice such a reader to a more broad-minded point of view. Thus Chapter 1 consists of a predigested summary of the topology required to understand what is happening in Chapter 2.

In Chapter 2, we give an overview of the algebraic topology and K-theory associated with cohomology of groups, and especially the extraordinary work of Quillen which has led to his definition of the higher algebraic K-groups of a ring.

The algebraic side of the cohomology of groups mirrors the topology, and we have always tried to give algebraic proofs of algebraic theorems. For example, in Chapter 3 you will find B. Venkov's topological proof of the finite generation of the cohomology ring of a finite group, while in Chapter 4 you will find L. Evens' algebraic proof. Also in Chapter 4, we give a detailed account of the construction of Steenrod operations in group cohomology using the Evens norm map, a topic usually treated from a topological viewpoint.

One of the most exciting developments in recent years in group cohomology is the theory of varieties for modules, expounded in Chapter 5. In a sense, this is the central chapter of the entire two volumes, since it shows how inextricably intertwined representation theory and cohomology really are.

I would like to record my thanks to the people, too numerous to mention individually, whose insights I have borrowed in order to write these volumes; who have pointed out infelicities and mistakes in the exposition; who have supplied me with quantities of coffee that would kill an average horse; and who have helped me in various other ways. I would especially like to thank Ken Brown for allowing me to explain his approach to induction theorems in I, Chapter 5; Jon Carlson for collaborating with me over a number of years, and without whom these volumes would never have been written; Ralph Cohen for helping me understand the free loop space and its rôle in cyclic homology (Chapter 2 of Volume II); Peter Webb for supplying me with an early copy of the notes for his talk at the 1986 Arcata conference on Representation Theory of Finite Groups, on which Chapter 6 of Volume II is based; David Tranah of Cambridge University Press for sending me a free copy of Tom Körner's wonderful book on Fourier analysis, and being generally helpful in

various ways you have no interest in hearing about unless you happen to be David Tranah.

There is a certain amount of overlap between this volume and my Springer lecture notes volume [**28**]. Wherever I felt it appropriate, I have not hesitated to borrow from the presentation of material there. This applies particularly to parts of Chapters 1, 4 and 5 of Volume I and Chapter 5 of Volume II.

THE SECOND EDITION. In preparing the paperback edition, I have taken the liberty of completely retypesetting the book using the enhanced features of LaTeX 2_ε, $\mathcal{A}\mathcal{M}\mathcal{S}$-LaTeX 1.2 and XY-pic 3.5. Apart from this, I have corrected those errors of which I am aware. I would like to thank the many people who have sent me lists of errors, particularly Bill Crawley–Boevey, Steve Donkin, Jeremy Rickard and Steve Siegel.

The most extensively changed sections are Section 2.2 and 3.1 of Volume I and Section 5.8 of Volume II, which contained major flaws in the original edition. In addition, in Section 3.1 of Volume I, I have changed to the more usual definition of Hopf algebra in which an antipode is part of the definition, reserving the term bialgebra for the version without an antipode. I have made every effort to preserve the numbering of the sections, theorems, references, and so on from the first edition, in order to avoid reference problems. The only exception is that in Volume I, Definition 3.1.5 has disappeared and there is now a Proposition 3.1.5. I have also updated the bibliography and improved the index. If you find further errors in this edition, please email me at djb@byrd.math.uga.edu.

Dave Benson, Athens, September 1997

Conventions and Notations.

- Maps will usually be written on the left. In particular, we use the left notation for conjugation and commutation: ${}^g h = ghg^{-1}$, $[g,h] = ghg^{-1}h^{-1}$, and ${}^g H = gHg^{-1}$.
- We write G/H to denote the action of G as a transitive permutation group on the left cosets of H.
- We write $H \leq_G K$ to denote that "H is G-conjugate to a subgroup of K". Similarly $h \in_G K$ means "h is G-conjugate to an element of K". Thus we write for example $\bigoplus_{g \in_G G}$ to denote a direct sum over conjugacy classes of elements of G.
- The symbol $\square$ denotes the end of a proof.
- We shall use the usual notations $O_p(G)$ for the largest normal p-subgroup of G, $O^p(G)$ for the smallest normal subgroup of G for which the quotient is a p-group, $G^{(\infty)}$ for the smallest normal subgroup of G for which the quotient is soluble, $\Phi(P)$ for the *Frattini subgroup* of a p-group P, i.e., the smallest normal subgroup for which the quotient is elementary abelian, $Z(G)$ for the centre of G, $\Omega_1(G)$ for the subgroup of an abelian p-group G generated by the elements of order p, and so on. The *p-rank* $r_p(G)$ is defined to be the maximal rank of an elementary abelian p-subgroup of G.
- If H and K are subgroups of a group G, then $\sum_{HgK}$ will denote a sum over a set of double coset representatives g of H and K in G.
- We shall write ${}_\Lambda M_\Gamma$ to denote that M is a Λ-Γ-bimodule, i.e., a left Λ-module which is simultaneously a right Γ-module in such a way that $(\lambda m)\gamma = \lambda(m\gamma)$ for all $\lambda \in \Lambda$, $m \in M$ and $\gamma \in \Gamma$.
- If G is a group of permutations on the set $\{1, \dots, n\}$ and H is another group, we write $G \wr H$ for the *wreath product*; namely the semidirect product of G with a direct product of n copies of H. Thus elements of $G \wr H$ are of the form $(\pi; h_1, \dots, h_n)$ with $\pi \in G$, $h_1, \dots, h_n \in H$ and multiplication given by
$$(\pi'; h'_1, \dots, h'_n)(\pi; h_1, \dots, h_n) = (\pi'\pi; h'_{\pi(1)}h_1, \dots, h'_{\pi(n)}h_n).$$
- If X is a set with a right G-action and Y is a set with a left G-action, then we write $X \times_G Y$ for the quotient of $X \times Y$ by the equivalence relation $(xg, y) \sim (x, gy)$ for all $x \in X$, $g \in G$, $y \in Y$.