CHAPTER 1

Background from algebraic topology

When we come to give a survey of the cohomology of groups in Chapter 2, we shall need quite a lot of elementary homotopy theory. For the convenience of the reader, we collect in this chapter some of the necessary topological background, and indicate where further details may be found.

1.1. Spaces of maps

First we recall a basic fact from general topology about spaces of maps. If X and Y are topological spaces, we write $\mathrm{Map}(X,Y)$ or Y^X for the space of (continuous) maps from X to Y with the compact-open topology. This is the topology for which the typical sub-basic open set is the set of maps taking a given compact set in X into a given open set in Y.

PROPOSITION 1.1.1. *If X and Y are Hausdorff and Y is locally compact, then the natural map*

$$\mathrm{Map}(X \times Y, Z) \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

sending f to the map f' defined by $f'(x)(y) = f(x,y)$ is a homeomorphism.

PROOF. See for example Hu [**128**, Section V.3]. □

This isomorphism is called the **exponential isomorphism**. This terminology becomes clearer if we write it in the form

$$Z^{Y\times X} \cong (Z^Y)^X.$$

We shall also work with spaces X with a *basepoint* x_0, which we shall denote by (X, x_0), and with pairs of spaces $A \subseteq X$ and a basepoint $x_0 \in A$, which we shall denote by (X, A, x_0). Maps of spaces should take the basepoint to the basepoint.

If (X, x_0) and (Y, y_0) are based spaces, we write $\mathrm{Map}_*(X,Y)$ for the subspace of $\mathrm{Map}(X,Y)$ consisting of those maps taking x_0 to y_0. This is a based space with the constant map as basepoint. If (Z, z_0) is another based space, then under the above correspondence, it is easy to see that the subspace of $\mathrm{Map}(X \times Y, Z)$ corresponding to the subspace

$$\mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z)) \subseteq \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

consists of those maps sending $(X \times y_0) \cup (x_0 \times Y)$ to z_0.

DEFINITION 1.1.2. *If* (X, x_0) *and* (Y, y_0) *are based spaces, we write* $X \vee Y$ *(*X **wedge** Y*) for the subspace* $(X \times y_0) \cup (x_0 \times Y)$ *of* $X \times Y$*, and* $X \wedge Y$ *(*X **smash** Y*) for the quotient space of* $X \times Y$ *formed by identifying all points of* $X \vee Y$ *to a single basepoint* $*$*.*

Thus we have the following:

PROPOSITION 1.1.3. *If* (X, x_0) *and* (Y, y_0) *are Hausdorff and* (Y, y_0) *is locally compact, then the natural map*

$$\mathrm{Map}_*(X \wedge Y, Z) \to \mathrm{Map}_*(X, \mathrm{Map}_*(Y, Z))$$

defined above is a homeomorphism. □

1.2. Homotopy groups

In this section we give an extremely compressed account of the homotopy groups π_n of a space. The interested reader is advised to refer to a standard source, for example Mosher and Tangora [**195**], Spanier [**247**], Switzer [**258**], Whitehead [**284**], for a more extensive account of this topic.

DEFINITION 1.2.1. *If* $f, f' : X \to Y$ *are (continuous) maps of topological spaces, then we say* f *is* **homotopic** *to* f' *(written* $f \simeq f'$*) if there exists a map* $F : X \times I \to Y$ *(where* I *is the unit interval* $[0,1]$*) such that* $F(x, 0) = f(x)$ *and* $F(x, 1) = f'(x)$*. It is clear that homotopy is an equivalence relation on maps. We write* $[X; Y]$ *for the homotopy classes of maps from* X *to* Y*.*

We say X *and* Y *are* **homotopy equivalent** *if there are maps* $f : X \to Y$ *and* $f' : Y \to X$ *such that the composites are homotopic to the identity maps* $f \circ f' \simeq \mathrm{id}_Y$ *and* $f' \circ f \simeq \mathrm{id}_X$*.*

We say X *is* **contractible** *if it is homotopy equivalent to a single point. In particular, note that a contractible space must be non-empty.*

We write $[X, x_0; Y, y_0]$ for homotopy classes of maps respecting basepoints. The homotopies should respect basepoints in the sense that $F(x_0 \times I) = y_0$. If A is a subspace of X containing x_0 and B is a subspace of Y containing y_0, we write $f : (X, A, x_0) \to (Y, B, y_0)$ to indicate that $f(A) \subseteq B$. We write $[X, A, x_0; Y, B, y_0]$ for homotopy classes of maps $f : X \to Y$ such that $f(A) \subseteq B$ and $f(x_0) = y_0$. Of course, the homotopies $F : X \times I \to Y$ should also have the property that $F(A \times I) \subseteq B$ and $F(x_0 \times I) = y_0$. If F is constant on A, so that f and f' agree on A, we say that f is homotopic to f' **relative to** A.

DEFINITION 1.2.2. *The* **homotopy groups** *of a based space* (X, x_0) *are defined to be*

$$\pi_n(X, x_0) = [S^n, s_0; X, x_0]$$

where (S^n, s_0) *is an* n*-sphere with basepoint.*

Thus for example $\pi_1(X, x_0)$ is the set of equivalence classes of paths from the basepoint to itself (**loops**), where two such are equivalent if one can be deformed to the other while keeping the two ends fixed. If X is path connected and $\pi_1(X, x_0) = 0$ then we say (X, x_0) is **simply connected**.

Since S^0 consists of two points, one of which is the basepoint, $\pi_0(X, x_0)$ is the set of path components of X.

GROUP STRUCTURE. The set $\pi_n(X, x_0)$ $(n \geq 1)$ can be given the structure of a group as follows. We regard (S^n, s_0) as an n-cube with its boundary identified to a single point, $(I^n, \dot{I}^n)$. Now to compose two elements $[f]$ and $[g]$ of $\pi_n(X, x_0)$, we "divide and stretch" along the first coordinate:

$$f * g : I^n \to X$$

$$(f * g)(s_1, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \dots, s_n) & \frac{1}{2} \leq s_1 \leq 1. \end{cases} \tag{1}$$

It is easy to check that if $f \simeq f'$ and $g \simeq g'$ then $f * g \simeq f' * g'$, so that this induces a well defined multiplication on $\pi_n(X, x_0)$.

PROPOSITION 1.2.3. *The above defined multiplication makes $\pi_n(X, x_0)$ into a group for $n \geq 1$, and an abelian group for $n \geq 2$.*

PROOF. The identity element is given by the constant map. The inverse in $\pi_n(X, x_0)$ of an element $[f]$ is given by $[f']$, where

$$f'(s_1, s_2, \dots, s_n) = f(1 - s_1, s_2, \dots, s_n).$$

The associative law corresponds to the "obvious" homotopy from $(f_1 * f_2) * f_3$ to $f_1 * (f_2 * f_3)$ given by

$$F((s_1, \dots, s_n), t) = \begin{cases} f_1(\frac{4s_1}{1+t}, s_2, \dots, s_n) & 0 \leq s_1 \leq \frac{1+t}{4} \\ f_2(4s_1 - 1 - t, s_2, \dots, s_n) & \frac{1+t}{4} \leq s_1 \leq \frac{2+t}{4} \\ f_3(\frac{4s_1 - 2 - t}{2 - t}, s_2, \dots, s_n) & \frac{2+t}{4} \leq s_1 \leq 1 \end{cases}$$

(Draw a diagram!) Similarly for $n \geq 2$, the commutative law corresponds to the following diagram:

$$\begin{array}{|c|c|} \hline f_1 & f_2 \\ & \\ \hline \end{array} \simeq \begin{array}{|c|c|} \hline f_1 & * \\ \hline * & f_2 \\ \hline \end{array} \simeq \begin{array}{|c|c|} \hline * & f_1 \\ \hline f_2 & * \\ \hline \end{array} \simeq \begin{array}{|c|c|} \hline f_2 & f_1 \\ & \\ \hline \end{array}$$

Here, the horizontal and vertical directions are the first and second coordinates in I^n, and the areas marked with an asterisk all go to the basepoint in X. □

REMARK. It is sometimes convenient to work with the (homotopy equivalent) space of maps from $[0, a] \times I^{n-1}$ to X (sending the boundary to x_0) rather than I^n to X, where a is regarded as a variable. Composition of a map from $[0, a] \times I^{n-1}$ and a map from $[0, b] \times I^{n-1}$ gives a map from $[0, a+b] \times I^{n-1}$. This composition has the advantage of being strictly associative rather than just homotopy associative. For $n = 1$, such maps are called **Moore loops** on X.

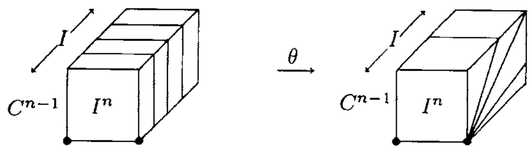


FIGURE 1. The map θ

DEFINITION 1.2.4. *The group* $\pi_1(X, x_0)$ *is called the* **fundamental group** *of the space* (X, x_0).

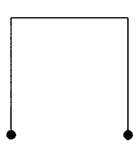
REMARK. The set of homotopy classes of paths in X with possibly different ends, where the homotopies are required to fix both ends of the path, form a groupoid. Namely, we can only compose one path with another if the endpoint of the first equals the starting point of the second. This groupoid is called the **fundamental groupoid** of X, and does not depend on choice of basepoint.

RELATIVE HOMOTOPY GROUPS. If (X, A, x_0) is a based pair, we define its **relative homotopy groups** to be

$$\pi_n(X, A, x_0) = [D^n, S^{n-1}, s_0; X, A, x_0] = [I^n, \dot{I}^n, C^{n-1}; X, A, x_0].$$

Here D^n is the n dimensional disc, with boundary S^{n-1}, and $C^{n-1} = \dot{I}^n \setminus \mathring{I}^{n-1}$ is the boundary of I^n with the interior of the face I^{n-1} (with the *last* coordinate zero) removed. This subspace is identified to a point and regarded as the basepoint.

For example if $n = 2$ then

$C^{n-1} =$ 

PROPOSITION 1.2.5. *Suppose* $\pi_n(X, A, x_0) = 0$. *Then any map*

$$f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$$

is homotopic relative to S^{n-1} *(i.e., via a homotopy which is constant on* S^{n-1}*) to a map* f' *with* $f'(D^n) \subseteq A$.

PROOF. Regard f as a map $(I^n, \dot{I}^n, C^{n-1}) \to (X, A, x_0)$. There is a homotopy $F : I^n \times I \to X$ with $F(u, 0) = f(u)$ and $F(u, 1) = x_0$ for all $u \in I^n$, $F(u, t) \in A$ for all $u \in \dot{I}^n$, and $F(u, t) = x_0$ for all $u \in C^{n-1}$. Composing F with the map $\theta : I^n \times I \to I^n \times I$ given in Figure 1 we obtain the required homotopy from f to a suitable f'. □

For $n \geq 2$, we can give $\pi_n(X, A, x_0)$ a multiplication according to Formula 1.

PROPOSITION 1.2.6. *The set $\pi_n(X, A, x_0)$ is a pointed set (i.e., a set with a distinguished element, namely the constant map at the basepoint) for $n = 1$, a group for $n \geq 2$, and an abelian group for $n \geq 3$.* □

WARNING. There is no excision in homotopy, so that in general

$$\pi_n(X, A, x_0) \neq \pi_n(X/A, A/A).$$

BOUNDARY MAP. If $f : (I^n, \dot{I}^n, C^{n-1}) \to (X, A, x_0)$ then we define

$$\partial_n(f) = f|_{I^{n-1}} : (I^{n-1}, \dot{I}^{n-1}) \to (A, x_0)$$

(note that $I^{n-1} \subseteq \dot{I}^n$ and $C^{n-1} \cap I^{n-1} = \dot{I}^{n-1}$). It is easy to check that if $f \simeq f'$ as maps of based pairs then $\partial_n(f) \simeq \partial_n(f')$ as maps of based spaces, so that ∂_n induces a well defined map

$$\partial_n : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \qquad (n \geq 1).$$

LONG EXACT SEQUENCE.

PROPOSITION 1.2.7. *Given a based pair (X, A, x_0), the natural maps $i : (A, x_0) \hookrightarrow (X, x_0)$ and $j : (X, x_0) \to (X, A, x_0)$ together with the boundary map defined above give rise to a long exact sequence*

$$\cdots \xrightarrow{j_*} \pi_{n+1}(X, A, x_0) \xrightarrow{\partial_*} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial_*} \cdots$$

REMARK. Exactness of the above sequence should be interpreted as follows. For $n \geq 1$, it is an exact sequence of groups, and the image of $\pi_2(X, A, x_0)$ lies in the centre of $\pi_1(A, x_0)$. For $n = 0$, these are only pointed sets (sets with a distinguished basepoint). The sequence is exact everywhere in the sense that the kernel of each map (pre-image of the basepoint) is the image of the previous map. Also, the map $\pi_1(X, A, x_0) \to \pi_0(A, x_0)$ extends to an action of $\pi_1(X, A, x_0)$ on $\pi_0(A, x_0)$, and elements are in the same orbit if and only if they have the same image in $\pi_0(X, x_0)$.

PROOF. We shall assume that $n \geq 2$ for the proof, and leave the reader to make the necessary changes for $n = 0$ and 1. There are six separate checks to be made here.

$j_* \circ i_* = 0$: If $f : (I^n, \dot{I}^n, C^{n-1}) \to (X, A, x_0)$ with $f(I^n) \subseteq A$ then we have a homotopy $F : I^n \times I \to X$ given by $F(t_1, \dots, t_n, t) = f(t_1, \dots, t_{n-1}, t + (1-t)t_n)$, showing that $[f] = 0$ in $\pi_n(X, A, x_0)$.

$\mathrm{Ker}(j_*) \subseteq \mathrm{Im}(i_*)$: If $f : (I^n, \dot{I}^n) \to (X, x_0)$ and $j_*[f] = 0$ then there is a homotopy $F : I^n \times I \to X$ such that $F(u, 0) = f(u)$ and $F(u, 1) = x_0$ for all $u \in I^n$, $F(u, t) \in A$ for all $u \in \dot{I}^n$, and $F(u, t) = x_0$ for all $u \in C^{n-1}$. Compose F with the map $\theta : I^n \times I \to I^n \times I$ given in Figure 1, to obtain a homotopy from f to a map $(I^n, \dot{I}^n) \to (A, x_0)$.

$i_* \circ \partial_* = 0$: If $f : (I^{n+1}, \dot{I}^{n+1}, C^n) \to (X, A, x_0)$ then $i_*\partial_*[f] = [f|_{I^n}]$. The map f provides a homotopy from $f|_{I^n}$ to the constant map at x_0.

$\mathrm{Ker}(i_*) \subseteq \mathrm{Im}(\partial_*)$: If $f : (I^n, \dot{I}^n) \to (A, x_0)$ with $i_*[f] = 0$ in $\pi_n(X, x_0)$, then there is a homotopy $F : (I^n \times I, \dot{I}^n \times I) \to (X, x_0)$ with $F(u, 0) = f(u)$ and $F(u, 1) = x_0$. Regarding F as a map $I^{n+1} \to X$, $[F]$ is an element of $\pi_{n+1}(X, A, x_0)$ with $\partial_*[F] = [f]$.

$\partial_* \circ j_* = 0$: If $f : (I^n, \dot{I}^n) \to (X, x_0)$ then $\partial_* j_*[f] = [f|_{I^{n-1}}]$. Since f sends I^{n-1} to x_0, $\partial_* j_*[f] = 0$.

$\mathrm{Ker}(\partial_*) \subseteq \mathrm{Im}(j_*)$: For this, we need the following fact. Given any map from $\dot{I}^n \times I \,\cup\, I^n \times \{0\}$ to X, we may extend it to a map from $I^n \times I$ to X. In other words, any partial homotopy may be extended to a homotopy. We express this by saying that the pair $(I^n, \dot{I}^n)$ has the **homotopy extension property** (HEP) with respect to X. A space which has the homotopy extension property with respect to all spaces X is called a **cofibration**. The reason why $(I^n, \dot{I}^n)$ is a cofibration is because there is a **retraction** $I^n \times I \to \dot{I}^n \times I \,\cup\, I^n \times \{0\}$, i.e., a map whose composite with the inclusion is the identity map on $\dot{I}^n \times I \,\cup\, I^n \times \{0\}$.

Now if $f : (I^n, \dot{I}^n, C^{n-1}) \to (X, A, x_0)$ with $\partial_*[f] = [f|_{I^{n-1}}] = 0$, then there is a homotopy $F : (I^{n-1} \times I, \dot{I}^{n-1} \times I) \to (A, x_0)$ with $F(u, 0) = f(u)$ and $F(u, 1) = x_0$. Thus there is a partially defined homotopy

$$\begin{aligned} \dot{I}^n \times I \,\cup\, I^n \times 0 &\to X \\ (u,t) \in I^{n-1} \times I &\mapsto F(u,t) \\ (u,t) \in (\dot{I}^n \setminus I^{n-1}) \times I &\mapsto x_0 \\ (u,t) \in I^n \times 0 &\mapsto f(u). \end{aligned}$$

Extend to a homotopy $I^n \times I \to X$, and restrict to $I^n \times 1$ to obtain a map $\alpha : I^n \to X$ with $f \simeq j(\alpha)$, so that $[f] = j_*[\alpha]$. □

Action of π_1 on π_n, and invariance of basepoint. If x_0 and x_1 are two different basepoints in X, and $\omega : I \to X$ is a path with $\omega(0) = x_0$ and $\omega(1) = x_1$, we define a map

$$\omega^* : \pi_n(X, x_1) \to \pi_n(X, x_0)$$

as follows. Given a map $f : (S^n, s_0) \cong (D^n, S^{n-1}) \to (X, x_1)$ we define

$$\omega^*(f) : (D^n, S^{n-1}) \to (X, x_0)$$

$$u \mapsto \begin{cases} f(2u) & \text{if } 0 \le |u| \le \frac{1}{2} \\ \omega(2 - 2|u|) & \text{if } \frac{1}{2} \le |u| \le 1 \end{cases}$$

where $|u|$ is the distance from the origin to u in $D^n \subseteq \mathbb{R}^n$, and $2u$ is calculated inside the vector space $\mathbb{R}^n$. In words, the part of the disk from radius 1 to radius $\frac{1}{2}$ is sent along the path ω. The part of the disk with radius at most $\frac{1}{2}$ is homeomorphic to the whole disk, and is sent into X by composing the original map with this homeomorphism.

If $f \simeq f'$ then $\omega^*(f) \simeq \omega^*(f')$ so that this map is well defined.

Similarly if x_0, $x_1 \in A \subseteq X$ are two different basepoints in (X, A), and $\omega : I \to A$ is a path with $\omega(0) = x_0$ and $\omega(1) = x_1$, we define an isomorphism

$$\omega^* : \pi_n(X, A, x_1) \to \pi_n(X, A, x_0)$$

as follows. Given a map $f : (D^n, S^{n-1}, (1, 0, \dots, 0)) \to (X, A, x_0)$, the map $\omega^*(f) : (D^n, S^{n-1}, (1, 0, \dots, 0)) \to (X, A, x_1)$ is given by sending $(x_0, \dots, x_n)$ to $\omega(x_0)$ if $0 \le x_0 \le 1$. We choose a homeomorphism between the half of the disk with $x_0 \le 0$ with the points with $x_0 = 0$ identified to a single point $*$, and the entire disk D^n with basepoint $(1, 0, \dots, 0)$. We compose this homeomorphism with the original map f to define $\omega^*(f)$ for $x_0 \le 0$.

Again if $f \simeq f'$ then $\omega^*(f) \simeq \omega^*(f')$, so that this map is well defined.

PROPOSITION 1.2.8. *If ω, $\omega' : I \to X$ are homotopic relative to $\{0, 1\}$, then $\omega^* = (\omega')^* : \pi_n(X, x_1) \to \pi_n(X, x_0)$. For $x_0 = x_1$, this constitutes a group action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.*

If $\omega \simeq \omega' : I \to A$ then $\omega^* = (\omega')^* : \pi_n(X, A, x_1) \to \pi_n(X, A, x_0)$. For $x_0 = x_1$, this constitutes a group action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$.

The long exact homotopy sequence

$$\cdots \to \pi_{n+1}(X, A, x_0) \to \pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \cdots$$

is a long exact sequence of $\pi_1(A, x_0)$-modules.

PROOF. This is a long but straightforward exercise in keeping your head. See Spanier [**247**, Section 7.3] for the details. □

NOTATION. We write $\pi'_n(X, x_0)$ for the quotient of $\pi_n(X, x_0)$ obtained by identifying $\omega^*(f)$ with f for each $[\omega] \in \pi_1(X, x_0)$, and $\pi'_n(X, A, x_0)$ for the quotient of $\pi_n(X, A, x_0)$ obtained by identifying $\omega^*(f)$ with f for each $[\omega] \in \pi_1(A, x_0)$. Note that the action of $\pi_1(X, x_0)$ on itself is by conjugation, so that $\pi'_1(X, x_0)$ is its abelianisation.

We say that (X, x_0) is a **simple space** if $\pi_1(X, x_0)$ acts trivially on $\pi_n(X, x_0)$ for all n, so that $\pi'_n(X, x_0) = \pi_n(X, x_0)$, and in particular $\pi_1(X, x_0)$ is abelian.

Elements of $\pi_n(X, x_0)$ may be thought of as obstructions to extending maps $S^n \to X$ to maps $D^{n+1} \to X$ as shown by the following proposition:

PROPOSITION 1.2.9. *Suppose X is path connected and $\pi_n(X, x_0) = 0$. Then any map $f : S^n \to X$ may be extended to a map $\tilde{f} : D^{n+1} \to X$.*

PROOF. Since X is path connected, $\pi_n(X, f(s_0)) = 0$, so the map $f : (S^n, s_0) \to (X, f(s_0))$ is homotopic to the constant map at $f(s_0)$. Such a homotopy is a map $S^n \times I \to X$ sending $S^n \times 0 \cup s_0 \times I$ to s_0 and equal to f on $S^n \times 1$. Since $(S^n \times I, S^n \times 0 \cup S^n \times 1 \cup s_0 \times I, S^n \times 0 \cup s_0 \times I) \cong (D^{n+1}, S^n, s_0)$ the proposition follows. □

The theory of obstructions was developed by Eilenberg and others, and an account of it may be found, for example, in Mosher and Tangora [**195**, Chapter 1].

Loop spaces. There is an alternative approach to homotopy groups based on the concept of a loop space.

Definition 1.2.10. *We define the* **loop space** ΩX *of a based space* X *to be*

$$\Omega X = \mathrm{Map}_*(S^1, X),$$

with the constant path at x_0 *(written* $*$*) as basepoint.*

Now a path in ΩX is the same thing as a homotopy between loops in X (by the correspondence given in Proposition 1.1.1), and so we have

$$\pi_1(X, x_0) \cong \pi_0(\Omega X, *).$$

By Proposition 1.1.3, we have

$$\Omega^2 X = \mathrm{Map}_*(S^1, \mathrm{Map}_*(S^1, X)) \cong \mathrm{Map}_*(S^1 \wedge S^1, X) \cong \mathrm{Map}_*(S^2, X).$$

Continuing in this way, and using the fact that $S^{n-1} \wedge S^1 \cong S^n$, we have

$$\Omega^n X \cong \mathrm{Map}_*(S^n, X).$$

A path in $\Omega^n X$ is the same as a homotopy between maps $S^n \to X$, and so we have proved the following:

Proposition 1.2.11. $\pi_n(X, x_0) \cong \pi_0(\Omega^n X, *)$. □

In a similar way, if (X, A, x_0) is a based pair, we define $P(X, A, x_0)$ to be the space of paths in X beginning at x_0 and ending in A. By a similar argument to the above, we have the following:

Proposition 1.2.12. $\pi_n(X, A, x_0) \cong \pi_0(\Omega^{n-1} P(X, A, x_0), *)$. □

For further details see for example Switzer [**258**, Chapter 3], where the long exact sequence (Proposition 1.2.7) is developed from this point of view.

Exercise. The (reduced) **suspension** SX of a space X is defined to be $S^1 \wedge X$. The nth suspension of X is $S^n X = S \cdots SX = S^n \wedge X$. Show that if X is Hausdorff then there is a natural homeomorphism

$$\mathrm{Map}_*(SX, Y) \to \mathrm{Map}_*(X, \Omega Y).$$

Deduce that there is a natural bijection

$$[S^n X; Y] \cong [X; \Omega^n Y].$$

Thus S and Ω are adjoint functors.

1.3. The Hurewicz theorem

For an arbitrary topological space, we have the **singular homology** $H_p(X;R)$ with coefficients in a commutative ring R, defined as the homology of the singular chain complex $\Delta_*(X,A;R) = \Delta_*(X;R)/\Delta_*(A;R)$, where $\Delta_p(X;R) = \Delta_p(X) \otimes R$ is the free R-module on the singular p-simplices. A singular p-simplex in X is a (continuous) map $\Delta^p \to X$ where Δ^p is a standard p-simplex

$$\Delta^p = \{(x_0, \dots, x_p) \in \mathbb{R}^{p+1} \mid \sum x_i = 1 \text{ and each } x_i \geq 0\}.$$

For more details see for example Spanier [**247**]. Similarly **singular cohomology** $H^p(X;R)$ is the cohomology of the cochain complex

$$\Delta^p(X,A;R) = \Delta^p(X,R)/\Delta^p(A;R),$$

where $\Delta^p(X;R) = \mathrm{Hom}(\Delta_p(X),R)$. In case $R = \mathbb{Z}$, we write $H_p(X)$ and $H^p(X)$.

Recall that if (X,x_0) is a based space then $\pi_n(X,x_0)$ is defined to be the group of homotopy classes of maps from (S^n,s_0) to (X,x_0). If $f:(S^n,s_0) \to (X,x_0)$ is such a map, then we have an induced map in homology

$$f_* : H_n(S^n) \to H_n(X)$$

which only depends on the homotopy class $[f] \in \pi_n(X,x_0)$. Since $H_n(S^n) \cong \mathbb{Z}$, we can define

$$h_n([f]) = f_*(1) \in H_n(X)$$

to obtain a well defined map

$$h_n : \pi_n(X,x_0) \to H_n(X)$$

called the **Hurewicz map**.

Similarly if $[f] \in \pi_n(X,A,x_0)$, then $[f]$ is represented by a map

$$f : (D^n, S^{n-1}, s_0) \to (X,A,x_0).$$

This induces a map

$$f_* : H_n(D^n,S^{n-1}) \to H_n(X,A),$$

and since $H_n(D^n,S^{n-1}) \cong \mathbb{Z}$ we can define

$$h_n([f]) = f_*(1) \in H_n(X,A)$$

to obtain a well defined Hurewicz map

$$h_n : \pi_n(X,A,x_0) \to H_n(X,A).$$

PROPOSITION 1.3.1. *If $n \geq 1$ then $h_n : \pi_n(X,x_0) \to H_n(X)$ is a group homomorphism. If ω is a path in X then $h_n(\omega^*(f)) = h_n(f)$ (see Section 1.2 for notation), so that we have a well defined map $h_n : \pi_n'(X,x_0) \to H_n(X)$.*

If $n \geq 2$ then $h_* : \pi_n(X,A,x_0) \to H_n(X,A)$ is a group homomorphism. If ω is a path in A then $h_n(\omega^*(f)) = h_n(f)$, so that we have an induced map $h_n : \pi_n'(X,A,x_0) \to H_n(X,A)$.

PROOF. See Spanier [**247**, Section 7.4]. □

If we wish to compare the maps h_n for different values of n, we must choose the identifications $H_n(S^n) \cong \mathbb{Z}$ and $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ in a consistent way, since there are two possible choices for each n. Since $(S^n, s_0) \cong (I^n, \dot{I}^n)$ and $(D^n, S^{n-1}, s_0) \cong (I^n, \dot{I}^n, C^{n-1})$, we may do this inductively as follows.

$$\begin{array}{ccccccc}
H_{n+1}(I^{n+1}, C^n) & \longrightarrow & H_{n+1}(I^{n+1}, \dot{I}^{n+1}) & \xrightarrow{\partial_*} & H_n(\dot{I}^{n+1}, C^n) & \longrightarrow & H_n(I^{n+1}, C^n) \\
\| & & & & \cong\uparrow j_* & & \| \\
0 & & & & H_n(I^n, \dot{I}^n) & & 0
\end{array}$$

If z_n is our choice of generator for $H_n(I^n, \dot{I}^n)$ then we let $z_{n+1} = \partial_*^{-1} j_*(z_n)$.

With these choices of identifications, we have the following theorem:

THEOREM 1.3.2. *Given a based pair* (X, A, x_0) *we have a commutative diagram*

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & \pi_{n+1}(X, A, x_0) & \longrightarrow & \pi_n(A, x_0) & \longrightarrow & \pi_n(X, x_0) & \longrightarrow & \pi_n(X, A, x_0) & \longrightarrow \cdots \\
& \downarrow h_{n+1} & & \downarrow h_n & & \downarrow h_n & & \downarrow h_n & \\
\cdots \longrightarrow & H_{n+1}(X, A) & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow \cdots
\end{array}$$

PROOF. See Spanier [**247**, Section 7.4]. □

THEOREM 1.3.3 (Absolute Hurewicz theorem). *Suppose that* $\pi_i(X, x_0) = 0$ *for* $i < n$. *Then* $H_i(X) = 0$ *for* $0 < i < n$, *and* $h_n : \pi'_n(X, x_0) \to H_n(X)$ *is an isomorphism. Note that if* $n > 1$, $\pi'_n(X, x_0) = \pi_n(X, x_0)$, *while* $\pi'_1(X, x_0) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$, *the abelianisation of* $\pi_1(X, x_0)$.

THEOREM 1.3.4 (Relative Hurewicz theorem). *Suppose that* A *is path connected and*

$$\pi_i(X, A, x_0) = 0 \text{ for } i < n.$$

Then $H_i(X, A) = 0$ *for* $i < n$, *and* $h_n : \pi'_n(X, A, x_0) \to H_n(X, A)$ *is an isomorphism.*

PROOF. The above two theorems are proved simultaneously by induction, together with a third theorem called the "homotopy addition theorem". For the details, see Spanier [**247**, Section 7.5]. □

Note that the converse of the Hurewicz theorem is false. It is not hard to construct spaces (X, x_0) with $\pi_i(X, x_0)$ non-zero for infinitely many different $i > 0$, but with $H_i(X) = 0$ for all $i > 0$. However, the following theorem is easy to deduce from the absolute Hurewicz theorem.

THEOREM 1.3.5. *Suppose* $\pi_1(X, x_0) = 0$ *and* $H_i(X) = 0$ *for* $0 < i < n$. *Then* $\pi_i(X, x_0) = 0$ *for* $0 < i < n$, *and* $h_n : \pi_n(X, x_0) \to H_n(X)$ *is an isomorphism.* □