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Foreword

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1 Introduction

This volume is an integrated collection of papers working out several new directions of research on 3-folds under the unifying theme of *explicit birational geometry*. Section 3 summarises briefly the contents of the individual papers.

Mori theory is a conceptual framework for studying minimal models and the classification of varieties, and has been one of the main areas of progress in algebraic geometry since the 1980s. It offers new points of view and methods of attacking classical problems, both in classification and in birational geometry, and it raises many new problem areas. While birational geometry has inspired the work of many classical and modern mathematicians, such as L. Cremona, G. Fano, Hilda Hudson, Yu. I. Manin, V. A. Iskovskikh and many others, and while their results undoubtedly give us much fascinating experimental material as food for thought, we believe that it is only within Mori theory that this body of knowledge begins to acquire a coherent shape.

At the same time as providing adequate tools for the study of 3-folds, Mori theory enriches the classical world many times over with new examples and constructions. We can now, for example, work and play with hundreds of families of Fano 3-folds. From where we stand, we can see clearly that the classical geometers were only scratching at the surface, with little inkling of the gold mine awaiting discovery.

The theory of minimal models of surfaces works with nonsingular surfaces, and the elementary step it uses is Castelnuovo's criterion, which allows us to contract -1-curves (exceptional curves of the first kind). A chain of such contractions leads us to a minimal surface S, either $\mathbb{P}^2$ or a scroll over a curve, or a surface with K_S numerically nonnegative (now called *nef*, see 2.2 below). These ideas were well understood by Castelnuovo and Enriques a century ago, and are so familiar that most people take them for granted. However, their higher dimensional generalisation was a complete mystery until the late 1970s, and may still be hard to grasp for newcomers to the field. It involves a suitable category of mildly singular projective varieties, and the crucial new ingredient of *extremal ray* introduced by Mori around 1980. As

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we discuss later in this foreword, extremal rays provide the elementary steps of the minimal model program (the divisorial contractions and flips of the Mori category that generalise Castelnuovo's criterion) and also the definition of Mori fibre space (that generalise $\mathbb{P}^2$ and the scrolls), our primary object of interest.

Higher dimensional geometry, like most other areas of mathematics, is marked by creative tensions between abstract and concrete on the one hand, general and special on the other. Contracting a -1-curve on a surface is a concrete construction, whereas a Mori extremal ray and its contraction is abstract (compare Remark 2.4.1). The "general" tendency in the classification of varieties, exemplified by the work of Iitaka, Mori, Kollár, Kawamata and Shokurov, includes things like Iitaka–Kodaira dimension, cohomological methods, and the minimal model program in substantial generality. The "special" tendency, exemplified by Hudson, Fano, Iskovkikh, Manin, Pukhlikov, Mori and ourselves, includes the study of special cases, for their own sake, and sometimes without hope of ever achieving general status.

By *explicit*, we understand a study that does not rest after obtaining abstract existence results, but that goes on to look for a more concrete study of varieties, say in terms of equations, that can be used to bring out their geometric properties as clearly as possible. For example, the list of Du Val singularities by equations and Dynkin diagrams is much more than just an abstract definition or existence result, and can be used for all kinds of purposes. This book initiates a general program of explicit birational geometry of 3-folds (compare Section 5). On the whole, our activities do not concern themselves with 3-folds in full generality, but work under particular assumptions, for example, with 3-folds that are hypersurfaces, or have only terminal quotient singularities (see below). The advantage is that we can get a long way into current thinking on 3-folds while presupposing little in the way of technical background in Mori theory.

Treating 3-folds and contractions between them in complete generality would lead us of necessity into a number of curious and technically difficult backwaters; these include many research issues of great interest to us, but we leave them to more appropriate future publications (see however Section 5 below). Making the abstract machinery work in dimension ≥ 4 is another important area of current research, but the geometry of 4-folds is presumably intractable in the explicit terms that are our main interest here.

2 The Mori program

This section is a gentle introduction to some of the ingredients of the 3-fold minimal model program, with emphasis on the aspects most relevant to our current discussion. Surveys by Reid and Kollár [R1], [Kol1], [Kol2] also offer introductory discussions and different points of view on Mori theory. At a technically more advanced level, we also recommend a number of excellent (if somewhat less gentle) surveys: Clemens, Kollár and Mori [CKM], Kawamata, Matsuda and Matsuki [KMM], Kollár and Mori [KM], Mori [Mo] and Wilson [W].

2.1 Terminal singularities

It was understood from the outset that minimal models of 3-folds necessarily involve singular varieties (one reason why is explained in 2.5). The *Mori category* consists of projective varieties with *terminal singularities*; the most typical example is the cyclic quotient singularity $\frac{1}{r}(a, r-a, 1)$. Here a is coprime to r, and the notation means the quotient $\mathbb{C}^3/(\mathbb{Z}/r)$, where the cyclic group $\mathbb{Z}/r$ acts by

$$(x, y, z) \mapsto (\varepsilon^a x, \varepsilon^{r-a} y, \varepsilon z),$$

and ε is a primitive rth root of 1. The most common instance is $\frac{1}{2}(1,1,1)$, the cone on the Veronese surface. The effect of saying that this point is terminal is that if we first resolve it by blowing up, then run a minimal model program on the resolution, we will eventually need to contract down everything we've blown up, taking us back to the same singularity.

There are a few other classes of terminal singularities, including isolated hypersurface singularities such as $xy = f(z,t) \subset \mathbb{C}^4$, where $f(z,t) = 0$ is an isolated plane curve singularity, and a combination of hypersurface and quotient singularity, for example, the hyperquotient singularity obtained by dividing the hypersurface singularity $xy = f(z^r, t)$ by the cyclic group $\mathbb{Z}/r$ acting by $\frac{1}{r}(a, r-a, 1, 0)$. At some time you may wish to look through some sections of Reid [YPG] (especially Theorem 4.5) for a more formal treatment. But for most purposes, the cyclic quotient singularity $\frac{1}{r}(a, r-a, 1)$ is the main case for understanding 3-fold geometry, and if you bear this in mind, you will have little trouble understanding this book.

2.2 Theorem on the Cone

The *Mori cone* $\overline{\mathrm{NE}}\, X$ (see Figure 2.2.1) is probably the most profound and revolutionary of Mori's contributions to 3-folds. An n-dimensional projective variety X over $\mathbb{C}$ is a $2n$-dimensional oriented compact topological space, and its second homology group $H_2(X, \mathbb{R})$ is a finite dimensional real vector space. Every algebraic curve $C \subset X$ can be triangulated and viewed as an oriented 2-cycle, and thus has a homology class $[C] \in H_2(X, \mathbb{R})$. Then by definition $\overline{\mathrm{NE}}\, X$ is the closed convex cone in $H_2(X, \mathbb{R})$ generated by the classes

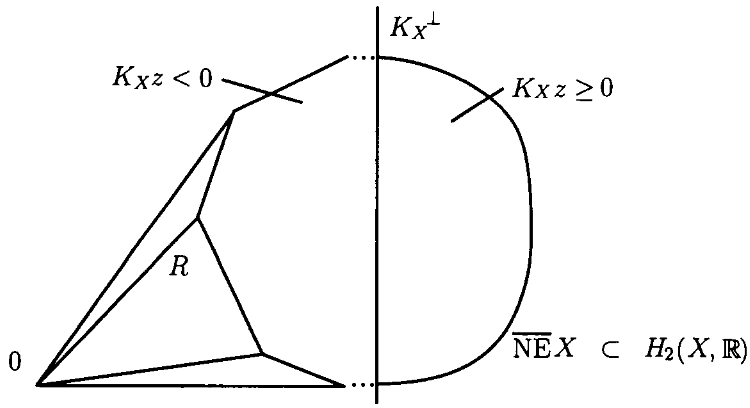

Figure 2.2.1: The Mori cone: $\overline{\mathrm{NE}}\,X$ is locally rational polyhedral in $K_X z < 0$

$[C]$ of algebraic curves $C \subset X$. You can think of this as follows: $H_2(X,\mathbb{R})$ is a property of the topological space X, whereas the structure of X as a projective algebraic variety provides the extra information of the Mori cone $\overline{\mathrm{NE}}\,X \subset H_2(X,\mathbb{R})$.

The shape of $\overline{\mathrm{NE}}\,X$ contains information about linear systems and embeddings $X \hookrightarrow \mathbb{P}^N$. Taking intersection number $D\cdot C$ with a divisor, or evaluating $\alpha \cap [C]$ with a cohomology class $\alpha \in H^2(X,\mathbb{R})$ (say, the first Chern class of a line bundle L) defines a linear form on $H_2(X,\mathbb{R})$. We say that D or α is *nef* if this linear form is ≥ 0 on $\overline{\mathrm{NE}}\,X$; that is, a divisor D is nef if $D \cdot C \geq 0$ for every curve $C \subset X$. Under an embedding, every algebraic curve must have positive degree; it is known that, under rather mild assumptions, X is projective if and only if $\overline{\mathrm{NE}}\,X$ is a genuine cone with a point.

To state Mori's theorem, we assume that the canonical divisor class K_X, (or equivalently, the first Chern class of the cotangent bundle) makes sense as a linear form on $H_2(X,\mathbb{R})$. This is a mild extra assumption on X, that certainly holds if X is nonsingular or has at worst quotient singularities. The theorem on the cone then says that $\overline{\mathrm{NE}}\,X$ is a rational polyhedral cone in the half-space of $H_2(X,\mathbb{R})$ on which K_X is negative. This theorem is particularly powerful for Fano varieties, defined by the condition that $-K_X$ is ample: for these, the entire cone $\overline{\mathrm{NE}}\,X$ is contained in $K_X z < 0$, so that $\overline{\mathrm{NE}}\,X$ is a finite rational polyhedral cone.

2.3 Extremal rays and the contraction theorem

Mori theory applies mainly to varieties with K_X not nef. This condition says that $K_X C < 0$ for some curve C, or that the part of the cone $\overline{\mathrm{NE}}\,X$ in the half-space $K_X z < 0$ is nonempty. Since this part of the cone is locally rational polyhedral, it follows that, if K_X is not nef, $\overline{\mathrm{NE}}\,X$ has at least one extremal ray R with $K_X \cdot R < 0$. Here an *extremal ray* is just a half-line $R = \mathbb{R}_+ z \subset \overline{\mathrm{NE}}\,X$ that is extremal in the sense of convex geometry (that is,

$$z_1, z_2 \in \overline{\mathrm{NE}}\,X \text{ and } z_1 + z_2 \in R \implies z_1, z_2 \in R).$$

Let $R \subset \overline{\mathrm{NE}}\,X$ be an extremal ray with $K_X \cdot R < 0$. Then there exists a contraction morphism

$$f_R \colon X \to Y,$$

characterised by the property that a curve $C \subset X$ is mapped to a point if and only if $C \in R$ (more precisely, the class of C). The morphism $f_R \colon X \to Y$ is called a *Mori contraction* or an *extremal contraction*. It is determined by the extremal ray R, and has categorical properties such as $-K_X$ relatively ample and $\rho(X/Y) = 1$ that turn out to be surprisingly strong: for example $-K_X$ ample puts us in a position where vanishing results based on Kodaira vanishing kill almost all the cohomology.

The cone and contraction theorems are proved in Kollár and Mori [KM]; on the whole, we can get by without reference to the technicalities of the proof, and you may prefer to take these results on trust for now.

2.4 Types of extremal rays

The next step is the case division on the dimension of the image Y and of the exceptional locus of the contraction morphism $f_R \colon X \to Y$, called the *classification of extremal rays* (or *rough classification*). The cases when the contraction $f_R \colon X \to Y$ has $\dim Y < \dim X$ lead to the definition of Mori fibre space and Fano varieties discussed in 2.6. In the other cases, we are dealing with birational modifications of X, and, as we see in 2.5, the aim is to proceed inductively towards a minimal model, as in the classical case of surfaces.

Remark 2.4.1 Note the contrast with the classical case: for surfaces, the thing we contract is a geometric locus. We find a -1-curve C and establish that it can be contracted in terms of a neighbourhood of C. In contrast, Mori theory in dimension ≥ 3 works primarily in terms of *categorical definitions* and *existence theorems*: the thing to be contracted is an extremal ray R of $\overline{\mathrm{NE}}\,X$ (the definition of which uses the totality of curves on X). The proof of the general theorems saying that R is contractible by a morphism f_R makes sophisticated use of numerical and cohomology vanishing properties of X.

The geometric nature of the contraction is only studied as a second step; even basic things such as the geometric locus that is contracted or even the dimension of the image cannot be anticipated. f_R may be birational, a proper fibre space, or the constant morphism to a point. This curious inversion of thinking is another of Mori's characteristic contributions to the subject, and the logic still comes as a surprise to anyone knowing a traditional treatment of the classification of surfaces. After all, Castelnuovo and Enriques could scarcely have guessed that (i) contracting a -1-curve, (ii) projecting a geometrically ruled surface to its base curve, and (iii) the constant map of $\mathbb{P}^2$ to a point would find a unified treatment as extremal contractions, and that this idea, however outlandish it might appear at first sight, would lay the foundations of all future work in classification.

2.5 Birational modifications: divisorial contractions, flips and the minimal model program

The extremal contractions that are most similar to contracting a -1-curve on a nonsingular surface (Castelnuovo's criterion) are the *divisorial contractions*. Here the case assumption is that $f_R\colon X \to Y$ is birational, and contracts a divisor of X to a locus of Y of codimension ≥ 2. The categorical properties of f_R then guarantee automatically that the exceptional locus of f_R is an *irreducible* divisor, and that Y has terminal singularities. This is the point at which terminal singularities force themselves on our attention: even if X is nonsingular, Y may be singular. Because Y is still in the Mori category, we can repeat the same game starting from Y.

The other birational case is when f_R is *small*, that is, every component of $\operatorname{Exc} f_R$ has codimension ≥ 2; in this case there cannot be any cohomology class in $H^2(X, \mathbb{R})$ that corresponds to the canonical divisor of Y, so that Y can *never* have terminal singularities. (If such a class existed, its pullback to X would coincide with K_X, which would then be numerically trivial on the fibres of f_R. This contradicts $-K_X$ ample, the defining property of a Mori extremal contraction.)

Because Y is no longer in the Mori category, the minimal model program cannot just continue inductively from Y. The subject was stuck at this point for a few years in the 1980s, before Mori proved the 3-fold flip theorem: there is a *flip*

$$\begin{array}{ccc} X & \overset{t_R}{\dashrightarrow} & X^+ \\ & \searrow \quad \swarrow & \\ & Y & \end{array} \qquad (2.5.1)$$

where $X^+ \to Y$ is another birational map from a 3-fold X^+, characterised by the property that K_{X^+} is ample over Y. In other words, the birational map $t_R\colon X \dashrightarrow X^+$ cuts out from X a finite number of curves on which K_X is negative, and in their place glues back into X^+ a finite number of curves on which K_{X^+} is positive. The definition of flip may seem somewhat obscure, but many nice attributes of X^+ follow from it; in particular, the morphism $X^+ \to Y$ is also small, and X^+ again has terminal singularities, so is in the Mori category. In dimension ≥ 4, the existence of the flip diagram (2.5.1) is called the *flip conjecture*; this seems to be one of the most intractable problems in the subject.

Divisorial contractions and Mori flips are the elementary steps in the Mori minimal model program. A sequence of these leads after a finite number of steps to a variety X', which is either a *minimal model*, that is, a variety with $K_{X'}$ nef, or a Mori fibre space $f\colon X' \to S$.

2.6 The definition of Mori fibre space

We now discuss the remaining cases in the classification of extremal rays, when the contraction $f_R\colon X \to Y$ maps to a smaller dimensional variety, that is, $\dim Y < \dim X$. Then f_R (or X itself) is called a *Mori fibre space* (Mfs). Note that, following Iitaka and Ueno, we say *fibre space* to mean a morphism $f\colon X \to Y$, often assumed to have connected fibres and Y normal, possibly with varying fibres, singular fibres, even fibres of different dimensions; this is not to be confused with the much stricter notion of fibre bundle.

The cases when Y is a surface and $X \to Y$ is a conic bundle (that is, the general fibre is a conic) or when Y is a curve and $X \to Y$ a fibre space of del Pezzo surfaces are the natural analogues of ruled surfaces. For the logical framework of Mori theory, we include in the definition of Mori fibre space the case that the contraction $f_R\colon X \to Y = \text{pt.}$ is the constant map to a point: then the morphism f_R is trivial, but its categorical properties include the fact that $-K_X$ is ample, and $\operatorname{Pic} X$ has rank 1. In this case X is called a *Fano 3-fold*; in contrast to the classical terminology, we allow X to be singular.

2.7 Biregular geometry versus birational geometry

The dividing line between biregular and birational geometry has changed through the generations, and is possibly still open to debate. The Italian school worked primarily in birational terms, and Zariski and Weil used birational ideas (at least in part) in setting up foundations for biregular geometry. The modern view, with scheme theory firmly established as the foundation, constructs birational geometry within this biregular framework. Thus, while the dichotomy between surfaces having nonvanishing plurigenera and ruled surfaces (or "adjunction terminates") is manifestly birational, we no longer think of it as the primary result of classification, but derive it from biregular results. This new view was instrumental in the success of Mori theory.

When we run a Mori minimal model program on a given 3-fold V, the end product is either a minimal model X with K_X nef, or a Mori fibre space, typically, a Fano 3-fold X or a conic bundle over a surface $X \to S$. The properties that define the 3-fold X are biregular in nature, so that we view X as a biregular construction. From this point of view, the proof of classification should also be considered a biregular activity, since the point is to prove that a given minimal 3-fold X has the right plurigenera and Kodaira dimension. Our conclusion is that birational geometry begins with the question of birational maps between different Mori fibre spaces.

3 What this book contains

This section discusses briefly the papers in this book, and their contribution to the above program of study. The papers are:

(1) K. Altmann: One-parameter families containing three-dimensional toric Gorenstein singularities

(2) J. Kollár: Nonrational covers of $\mathbb{P}^m \times \mathbb{P}^m$

(3) A. V. Pukhlikov: Essentials of the method of maximal singularities

(4) A. R. Iano-Fletcher: Working with weighted complete intersections

(5) A. Corti, A. Pukhlikov and M. Reid: Fano 3-fold hypersurfaces

(6) A. Corti: Singularities of linear systems and 3-fold birational geometry

(7) M. Reid: Twenty five years of 3-folds, an old person's view

Klaus Altmann's paper (1) is a study of the deformation theory of toric Gorenstein 3-fold singularities. It relates to the classification of 3-fold flips as follows: we know that any Mori flip diagram (2.5.1) can be obtained from a $\mathbb{C}^\times$ action on a 4-fold Gorenstein singularity $0 \in A$ by taking the quotient by the $\mathbb{C}^\times$ action in different interpretations – the so-called *variation of geometric invariant theory quotient*, see Dolgachev and Hu [DH], Reid [R2] and Section 5.3 below. Moreover, the general anticanonical divisor $S \in |-K_X|$ (the *general elephant*) is a surface with only Du Val singularities, according to Kollár and Mori [KM1], Theorem 1.7. Its inverse image in A is a $\mathbb{C}^\times$ cover $B \to S$, and is a hyperplane section $B \subset A$, so that A can be viewed as a 1-parameter deformation of B. It frequently happens that S is of type A_n, and then B is toric, so that Altmann's theory applies in many cases to give a classification of 3-fold flips. Altmann's previous work [Al] used the notion of Minkowski decomposition of polytopes to give a complete treatment of the deformation of isolated 3-fold toric Gorenstein singularities; in the present paper, he shows how to modify his method to the case of toric varieties having singularities in codimension 2.

János Kollár's paper (2) provides a new method of proving irrationality, adding to the known collection of rationally connected varieties that are not rational: finite covers of $\mathbb{P}^m \times \mathbb{P}^n$ with ramification divisor of large enough degree in one factor, and hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$ of large enough degree. His technique involves reduction to characteristic p, and a rather clever and surprising analysis of the stability of the tangent bundle in characteristic p. In fact, he proves the slightly more general structural property that these varieties are not even ruled. In the case of conic bundles, these results are spectacularly close to the conjectural bound for rationality (compare, for example, paper (2), Remark 1.2.1.1 with Corti's paper (6), 4.10 and 4.11). This provides the strongest confirmation to date of the conjectures on conic bundles, in a numerical range that is inaccessible to all other methods.

The papers (3)–(6) form a connected suite of papers around the subject of *birational rigidity*. The notion, discussed in more detail in Section 4.5 below, originates in the famous result of Iskovskikh and Manin [IM] that a non-singular quartic 3-fold $X_4 \subset \mathbb{P}^4$ has no birational maps to Fano varieties (other than isomorphisms to itself). Pukhlikov's paper (3) describes his important simplification and elaboration of Iskovskikh and Manin's treatment. This paper is partly based on notes of lectures given at the 1995–96 Warwick algebraic geometry symposium and the preprint, with its clear treatment of the Russian methods, strongly stimulated our collaboration in the joint paper (5). The different approach in Pukhlikov's papers also offers a useful ideological and practical counterweight to the methods of Corti's paper (6).

Our long joint paper (5) is the real heart of this book. In it, we carry out a substantial portion of a program of research on birational rigidity, treating the *famous 95* families of Fano 3-fold weighted hypersurfaces. We refer to Section 4.5 and the introduction to paper (5) for further discussion of birational rigidity.

Anthony Iano-Fletcher's paper (4) is a well written tutorial introduction to weighted projective spaces and their subvarieties. This paper has been available for many years as a Max Planck Institute preprint, and is widely quoted in the literature; it contains many very useful results and methods of calculation, including one derivation of the list of the famous 95 hypersurfaces, and thus forms an essential prerequisite for paper (5).

Corti's paper (6) contains a detailed introduction to the Sarkisov program. It develops and applies powerful new methods to quantify and analyse the singularities of linear systems, clarifying and providing technical alternatives to the methods initiated by Iskovskikh and Manin based on the study of the resolution graph. The new ideas are based on the Shokurov connectedness principle in log birational geometry, and seem to provide the most powerful currently known technique to exclude birational maps between Mori fibre spaces. The results are applied to give rigidity criteria for Mori fibre spaces in a number of cases, and our joint paper (5) also appeals to them for one or two technical points.

Reid's historical paper (7) is a *Heldenleben* that needs no introduction.

4 Mori theory and birational geometry

Following our introductory remarks on Mori theory in Section 2, we now give a brief introduction to our view of birational geometry, including the Sarkisov program and birational rigidity.

4.1 Fano-style projections

Fano based his treatment of the 3-folds $V_{2g-2} \subset \mathbb{P}^{g+1}$ for $g \geq 7$ on the idea of constructing a birational map by projection from a suitably chosen centre. Typically, the double projection of V from a line L involves a diagram

$$\begin{array}{ccc} V' & \dashrightarrow & V'' \\ \downarrow & & \downarrow \\ V & & W, \end{array} \tag{4.1.1}$$

where $V' \to V$ is the blowup of $L \subset V$, the map $V' \dashrightarrow V''$ flops the lines meeting L (in good cases, finitely many lines with normal bundle of type $(-1,-1)$), and $V'' \to W$ contracts the surface $E \subset V''$ swept out by conics meeting L to a curve $\Gamma \subset W$. Fano thought of the map $V \dashrightarrow W$ as the rational map defined by linear projection, and factoring it in biregular terms was not his primary concern.

For us, on the other hand, it is important to view Fano's projection as a general construction in the Mori category: $V' \to V$ is an extremal extraction, $V' \dashrightarrow V''$ a rational map that is an isomorphism in codimension 1 (in good cases, a composite of classic flops), and $V'' \to W$ the contraction of an extremal ray. All 4 of the varieties in (4.1.1) are in the Mori category, and the two morphisms are contractions of extremal rays.

Remark 4.1.1 We take the opportunity to clear up a possible source of confusion that occurs throughout the subject: in Fano's case, the *single* projection from L contracts the flopping lines by a morphism $V' \to \overline{V}$ to a variety $\overline{V}$ having (in good cases) only 3-fold ordinary double points; we think of $\overline{V}$ as the *midpoint* of the construction of the link $V \dashrightarrow W$. It is a Fano variety