K-theory, originally the study of vector bundles in topology, has become a powerful tool in the subject of operator algebras and has led to profound and unexpected applications throughout mathematics. This book develops K-theory and its deep bivariant version, Kasparov's KK-theory, from scratch to its most advanced aspects and describes important applications in topology and geometry. Numerous exercises ranging from elementary to research level supplement the text.

The second edition includes a new section on E-theory and coverage of other key recent developments in the subject, as well as more than sixty new and updated references.

From reviews of the first edition:
"This book gives a comprehensive survey of 'operator' K-theory or 'noncommutative' algebraic topology. Since its inception in the early 1970s, this field has grown rapidly, until a deep and elaborate machinery has evolved. This book is the first to consolidate this material and does an excellent job of presenting the path of least resistance to the key results while keeping the reader informed about the many important sidetracks."

– *Mathematical Reviews*

Mathematical Sciences Research Institute Publications

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K-Theory for Operator Algebras

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K-Theory for Operator Algebras

Second Edition

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For Martha and Gabriel

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Frontmatter
More information

PREFACE TO SECOND EDITION

I was pleased to learn that MSRI and Cambridge University Press have decided to issue a second edition of this book. The first edition sold out its press run rather quickly, and for the last several years I have had regular inquiries from colleagues and students about where to obtain a copy.

The first edition was successful because it filled a need: there was nothing available at the time which even remotely covered the subject satisfactorily. Even now, while there are many more references available for this and related subjects, there is no other comprehensive treatment of the topics covered in this book, and so the second edition will (I hope) still fill an important need. Some of the newer references I can recommend to the reader are: [Fillmore 1996] and [Murphy 1990] as general references for C^*-algebras, including some K-theory; [Davidson 1996] for a deep study of examples of important C^*-algebras, including many treated superficially in my book; [Wegge-Olsen 1993] for a leisurely treatment of basic K-theory in more detail than we have included here; [Higson 1990] for a detailed survey of KK-theory and its applications, written primarily for nonspecialists in operator algebras; [Rosenberg 1994] for an excellent treatment of algebraic K-theory; [Loday 1992] for cyclic homology/cohomology; and, of course, [Connes 1994] (which was already partially available at the time of the first edition)—its introduction gives a marvelous overview of the subject, and the book contains a vast supply of important applications. Several more books on operator K-theory and related subjects are forthcoming. There is also a compilation of all Mathematical Reviews on K-theory from 1940 to 1985 [Magurn 1985], and even a K-theory preprint archive on the web (www.math.uiuc.edu/K-theory/).

Time and mathematics have marched on since the book was completed, and the treatment of the book is somewhat out of date in many areas and badly out of date in a few. Despite my overly optimistic statement in the original preface that "it appears the basic theory has more or less reached a final form," the subject has continued to evolve. Even before the book was published, important new approaches and results of Cuntz and Skandalis made some sections somewhat obsolete. Since then, such advances as the E-theory of Connes and Higson have become core material for the subject. Profound new applications of K-theory and its generalizations, both within operator algebras (for example, the classification programs of Elliott *et al.* and of Kirchberg) and in topology and geometry (see [Connes 1994]) have attracted much attention.

In preparing the second edition, I had to make the most important decision right at the outset: how extensively to revise the book. If I were writing a book now from scratch, it would be very different from the first edition, taking into account both the evolution of the subject itself and of my understanding of it. I basically had three options:

(1) Extensively rewrite almost every section.
(2) Do a minimal revision, correcting errors, updating references, and adding a few comments here and there about the most important developments.
(3) Do something in between, rewriting some parts but leaving the essence of the manuscript intact.

I immediately rejected option (3): I was unlikely to be completely satisfied with the outcome, and if I was going to do that much work I should really do the job right and choose option (1). Option (1) was seriously considered, but ultimately rejected, partly because I did not feel I had the time or energy to do it properly, and partly because I think there are others far more qualified than I (some of whom are seriously considering a book of their own which I very much hope to see.) So option (2) was chosen. I felt that even though the treatment in the book might no longer be optimal, it could be relatively easily updated into a reference still of value. I have corrected those errors, gaps, and obscurities I was aware of, both typographical and mathematical (although I certainly would not be willing to assert that the book is now error-free!) I have also updated references, and added new comments and references where needed. I have tried to mention the most important new developments, usually in brief comments or exercises/problems; the reader should not infer that subjects treated only briefly here are not worthy of expanded coverage.

The one major addition to the book is a new section (§ 25) on E-theory. I felt that no contemporary book on K-theory would be complete without this topic, and it warranted more than a brief treatment since there is at present no other adequate reference. A few other additions of essential topics are scattered throughout the book.

I have taken care not to change the numbering of any section, paragraph, or result, in order not to make references ambiguous. The only exceptions to this are Theorem 11.4.2 and problem 16.4.8, which were erroneously numbered 11.4.1 and 16.4.7 respectively in the first edition. All added material has numbers not used in the first edition. This policy has occasionally led to new material being inserted in a less than optimal location, but I believe the advantages of the policy outweigh the disadvantages.

I am grateful to all those who have given or sent me comments or corrections to the first edition. Colleagues who have made substantive comments include P. Baum, J. Cuntz, E. Kirchberg, A. Kumjian, L. Lehmann, N. C. Phillips,

M. Rørdam, J. Rosenberg, and C. Schochet. Unfortunately, recollection of the source of some of the comments has been lost in time, and I apologize to anyone I missed in this list.

The first edition was produced with what was then a state-of-the-art word processing system, UNIX troff with eqn, and I received compliments on the appearance. Today the printing looks crude. I am grateful to Silvio Levy of MSRI who spent a great deal of time converting the troff files to TeX and making the book look good by current standards.

PREFACE TO FIRST EDITION

K-Theory has revolutionized the study of operator algebras in the last few years. As the primary component of the subject of "noncommutative topology," K-theory has opened vast new vistas within the structure theory of C^*-algebras, as well as leading to profound and unexpected applications of operator algebras to problems in geometry and topology. As a result, many topologists and operator algebraists have feverishly begun trying to learn each others' subjects, and it appears certain that these two branches of mathematics have become deeply and permanently intertwined.

Despite the fact that the whole subject is only about a decade old, operator K-theory has now reached a state of relative stability. While there will undoubtedly be many more revolutionary developments and applications in the future, it appears the basic theory has more or less reached a "final form." But because of the newness of the theory, there has so far been no comprehensive treatment of the subject.

It is the ambitious goal of these notes to fill this gap. We will develop the K-theory of Banach algebras, the theory of extensions of C^*-algebras, and the operator K-theory of Kasparov from scratch to its most advanced aspects. We will not treat applications in detail; however, we will outline the most striking of the applications to date in a section at the end, as well as mentioning others at suitable points in the text.

There is little in these notes which is new. They represent mainly a consolidation and integration of previous work. I have borrowed freely from the ideas and writings of others, and I hope I have been sufficiently conscientious in acknowledging the sources of my presentation within the text and in the notes at the end of sections. There are some places where I have presented new arguments or points of view to (hopefully) make the exposition cleaner or more complete.

These notes are an expanded and refined version of the lecture notes from a course I gave at the Mathematisches Institut, Universität Tübingen, West Germany, while on sabbatical leave during the 1982-83 academic year. I taught the course in an effort to learn the material of the later sections. I am grateful to the participants in the course, who provided an enthusiastic and critical audience: A. Kumjian, B. Kümmerer, M. Mathieu, R. Nagel, W. Schröder, J. Vazquez, M. Wolff, and L. Zsido; and to all the others in Tübingen who made my stay pleasant

and worthwhile. I am also grateful to the Alexander von Humboldt-Stiftung for their financial support through a Forschungsstipendium.

I have benefited greatly from numerous lectures and discussions at the Mathematical Sciences Research Institute, Berkeley, during the 1984–85 academic year. I am particularly indebted to P. Baum, L. Brown, A. Connes, J. Cuntz, R. Douglas, N. Higson, J. Kaminker, C. Phillips, M. Rieffel, J. Rosenberg, and C. Schochet for sharing their knowledge and insights.

In addition, I want to thank J. Cuntz, P. Julg, G. Kasparov, C. Phillips, M. Rieffel, J. Roe, D. Voiculescu, and especially J. Rosenberg, for taking the time to review a preliminary draft of the manuscript, pointing out a number of minor (and a few major) errors, and suggesting improvements.

I am also grateful to Ed Wishart, Jana Dunn, Bill Rainey, Mark Schank, and Ron Sheen for patiently helping me master the UNIX[1] system and managing to keep the UNR system operating long enough to produce the manuscript.

Since these notes are primarily written for specialists in operator algebras, we will assume familiarity with the rudiments of the theory of Banach algebras and C^*-algebras, such as can be found in the first part of [Dixmier 1969], [Pedersen 1979], or [Takesaki 1979]. Some of the sections, particularly later in the book, require more detailed knowledge of certain aspects of C^*-algebra theory.

Most of the notation we use will be standard, and will be explained as needed. Some basic notation used throughout: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ will denote the natural numbers, integers, and rational, real, and complex numbers respectively; $\mathbb{M}_n$ will denote the $n \times n$ matrices over C; $\mathbb{H}$ will denote a Hilbert space, separable and infinite-dimensional unless otherwise specified; and $\mathbb{B}(\mathbb{H})$ and $\mathbb{K}(\mathbb{H})$, or often just $\mathbb{B}$ and $\mathbb{K}$, will respectively denote the bounded operators and compact operators on $\mathbb{H}$. $\mathrm{diag}(x_1, \ldots, x_n)$ will denote the diagonal matrix with diagonal elements $x_1, \ldots, x_n$. If A and B are C^*-algebras, $A \otimes B$ will always denote the minimal (spatial) C^*-tensor product of A and B.

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[1] UNIX is a Trademark of Bell Laboratories.

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