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Radial Basis Functions: Theory and Implementations

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Preface

The radial basis function method for multivariate approximation is one of the most often applied approaches in modern approximation theory when the task is to approximate scattered data in several dimensions. Its development has lasted for about 25 years now and has accelerated fast during the last 10 years or so. It is now in order to step back and summarise the basic results comprehensively, so as to make them accessible to general audiences of mathematicians, engineers and scientists alike.

This is the main purpose of this book which aims to have included all necessary material to give a complete introduction into the theory and applications of radial basis functions and also has several of the more recent results included. Therefore it should also be suitable as a reference book to more experienced approximation theorists, although no specialised knowledge of the field is required. A basic mathematical education, preferably with a slight slant towards analysis in multiple dimensions, and an interest in multivariate approximation methods will be suitable for reading and hopefully enjoying this book.

Any monograph of this type should be self-contained and motivated and need not much further advance explanations, and this one is no exception to this rule. Nonetheless we mention here that for illustration and motivation, we have included in this book several examples of practical applications of the methods at various stages, especially of course in the Introduction, to demonstrate how very useful this new method is and where it has already attracted attention in real life applications. Apart from such instances, the personal interests of the author mean that the text is dominated by theoretical analysis. Nonetheless, the importance of applications and practical methods is underlined by the aforementioned examples and by the chapter on implementations. Since the methods are usually applied in more than two or three dimensions, pictures will unfortunately not help us here very much which explains their absence.

After an introduction and a summary in Chapter 2 of the types and results of analysis that are used for the radial basis functions, the order of the remaining chapters essentially follows the history of the development: the convergence analysis was first completed in the setting of gridded data, after some initial and seminal papers by Duchon, and then further results on scattered data and their convergence orders were found; subsequently, radial basis functions on compact support were studied, then (and now) efficient implementations and finally wavelets using radial basis functions are the most topical themes.

Few can complete a piece of work of this kind without helping hands from various people. In my case, I would like to thank first and foremost my teacher Professor Michael Powell who introduced me into radial basis function research at Cambridge some 17 years ago and has been the most valuable teacher, friend and colleague to me ever since. Dr David Tranah of Cambridge University Press suggested once that I should write this book and Dr Alan Harvey as an editor kept me on the straight and narrow thereafter. Dr Oleg Davydov, Mr Simon Hubbert, Dr Ulrike Maier, Professor Tomas Sauer and Professor Robert Schaback looked at various parts of the manuscript and gave many helpful suggestions. Finally, I would like to thank Mrs Marianne Pfister of ETH who has most expertly typed an early version of the manuscript and thereby helped to start this project.

At the time of proofreading this book, the author learnt about the death of Professor Will Light of Leicester University. Will's totally unexpected death is an irreplaceable less to approximation theory and much of what is being said in this book would have been unthinkable without his many contributions to and insights into the mathematics of radial basis functions.