Impact Mechanics

W. J. STRONGE
University of Cambridge

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK http://www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011-4211, USA http://www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

First published 2000

Printed in the United States of America

Typeface Times Roman 10.25/12.5 pt. *System* LaTeX 2_ε [TB]

*A catalog record for this book is available from
the British Library.*

Library of Congress Cataloging-in-Publication Data

Stronge, W. J. (William James), 1937–
Impact mechanics / W.J. Stronge.
p. cm.
ISBN 0-521-63286-2 (hb)
1. Impact. I. Title.
TA354 .S77 2000
620.1′125 – dc21
99-044947

ISBN 0 521 63286 2 hardback

Contents

CHAPTER 1

Introduction to Analysis of Low Speed Impact

> Philosophy is written in this grand book – I mean the universe – which stands continuously open to our gaze, but cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics and its characters are triangles, circles and other geometric figures, without which it is humanly impossible to understand a single word of it; without these one is wandering about in a dark labyrinth.
>
> Galileo Galilei, *Two New Sciences*, 1632

When a bat strikes a ball or a hammer hits a nail, the surfaces of two bodies come together with some relative velocity at an initial instant termed *incidence*. After incidence there would be interference or interpenetration of the bodies were it not for the interface pressure that arises in a small *area of contact* between the two bodies. At each instant during the contact period, the pressure in the contact area results in local deformation and consequent indentation; this indentation equals the interference that would exist if the bodies were not deformed.

At each instant during impact the interface or contact pressure has a resultant force of action or reaction that acts in opposite directions on the two colliding bodies and thereby resists interpenetration. Initially the force increases with increasing indentation and it reduces the speed at which the bodies are approaching each other. At some instant during impact the work done by the contact force is sufficient to bring the speed of approach of the two bodies to zero; subsequently, the energy stored during compression drives the two bodies apart until finally they separate with some relative velocity. For impact between solid bodies, the contact force that acts during collision is a result of the local deformations that are required for the surfaces of the two bodies to conform in the contact area.

The local deformations that arise during impact vary according to the incident relative velocity at the point of initial contact as well as the hardness of the colliding bodies. Low speed collisions result in contact pressures that cause small deformations only; these are significant solely in a small region adjacent to the contact area. At higher speeds there are large deformations (i.e. strains) near the contact area which result from plastic flow; these large localized deformations are easily recognizable, since they have gross manifestations such as cratering or penetration. In each case the deformations are consistent with the contact force that causes velocity changes in the colliding bodies. The normal impact speed required to cause large plastic deformation is between $10^2 \times V_Y$ and $10^3 \times V_Y$ where V_Y is the minimum relative speed required to initiate plastic yield in the softer body (for metals the normal incident speed at yield V_Y is of the order of 0.1 m s^{-1}). This text explains how the dynamics of low speed collisions are related to both local and global deformations in the colliding bodies.

1.1 Terminology of Two Body Impact

1.1.1 Configuration of Colliding Bodies

As two colliding bodies approach each other there is an instant of time, termed *incidence*, when a single *contact point* C on the surface of the first body B initially comes into contact with point C′ on the surface of the second body B′. This time $t = 0$ is the initial instant of impact. Ordinarily the surface of at least one of the bodies has a continuous gradient at either C or C′ (i.e., at least one body has a topologically smooth surface) so that there is a unique *common tangent plane* that passes through the coincident contact points C and C′. The orientation of this plane is defined by the direction of the normal vector **n**, a unit vector which is perpendicular to the common tangent plane.

Central or Collinear Impact Configuration:

If each colliding body has a center of mass G or G′ that is on the common normal line passing through C, the impact configuration is *collinear*, or central. This requires that the position vector $\mathbf{r}_C$ from G to C, and the vector $\mathbf{r}'_C$ from G′ to C′, both be parallel to the common normal line as shown in Fig. 1.1a:

$$\mathbf{r}_C \times \mathbf{n} = \mathbf{r}'_C \times \mathbf{n} = \mathbf{0}.$$

Collinear impact configurations result in equations of motion for normal and tangential directions that can be decoupled. If the configuration is not collinear, the configuration is *eccentric*.

Eccentric Impact Configuration:

The impact configuration is eccentric if at least one body has a center of mass that is off the line of the common normal passing through C as shown in Fig. 1.1b. This occurs if either

$$\mathbf{r}_C \times \mathbf{n} \neq \mathbf{0} \quad \text{or} \quad \mathbf{r}'_C \times \mathbf{n} \neq \mathbf{0}.$$

If the configuration is eccentric and the bodies are rough (i.e., there is a tangential force

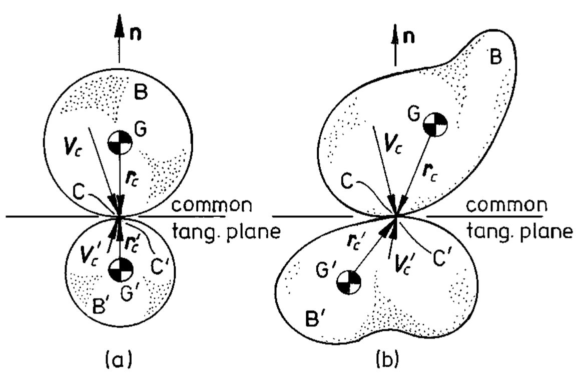


Figure 1.1. Colliding bodies B and B′ with (a) collinear and (b) noncollinear impact configurations. In both cases the angle of incidence is oblique; i.e. $\phi_0 \neq 0$.

of friction that opposes sliding), the equations of motion each involve both normal and tangential forces (and impulses). Thus eccentric impact between rough bodies involves effects of friction and normal forces that are not separable.

1.1.2 Relative Velocity at Contact Point

At the instant when colliding bodies first interact, the coincident contact points C and C′ have an initial or *incident relative velocity* $\mathbf{v}_0 \equiv \mathbf{v}(0) = \mathbf{V}_C(0) - \mathbf{V}'_C(0)$. The initial relative velocity at C has a component $\mathbf{v}_0 \cdot \mathbf{n}$ normal to the tangent plane and a component $(\mathbf{n} \times \mathbf{v}_0) \times \mathbf{n}$ parallel to the tangent plane; the latter component is termed *sliding*. The *angle of obliquity at incidence*, ψ_0, is the angle between the initial relative velocity vector $\mathbf{v}_0$ and the unit vector $\mathbf{n}$ normal to the common tangent plane,

$$\psi_0 \equiv \tan^{-1}\left(\frac{(\mathbf{n} \times \mathbf{v}_0) \times \mathbf{n}}{\mathbf{v}_0 \cdot \mathbf{n}}\right).$$

Direct impact occurs when in each body the velocity field is uniform and parallel to the normal direction. Direct impact requires that the angle of obliquity at incidence equals zero ($\psi_0 = 0$); on the other hand, *oblique impact* occurs when the angle of obliquity at incidence is nonzero ($\psi_0 \neq 0$).

1.1.3 Interaction Force

An interaction force and the impulse that it generates can be resolved into components normal and parallel to the common tangent plane. For particle impact the impulse is considered to be normal to the contact surface and due to short range interatomic repulsion. For solid bodies, however, contact forces arise from local deformation of the colliding bodies; these forces and their associated deformations ensure compatibility of displacements in the contact area and thereby prevent interpenetration (overlap) of the bodies. In addition a tangential force, *friction*, can arise if the bodies are *rough* and there is sliding in the contact area. Dry friction is negligible if the bodies are *smooth*.

Conservative forces are functions solely of the relative displacement of the interacting bodies. In an *elastic collision* the forces associated with attraction or repulsion are conservative (i.e. reversible); it is not necessary however for friction (a nonconservative force) to be negligible. In an *inelastic collision* the interaction forces (other than friction) are nonconservative, so that there is a loss of kinetic energy as a result of the cycle of compression (loading) and restitution (unloading) that occurs in the contact region. The energy loss can be due to irreversible elastic–plastic material behavior, rate-dependent material behavior, elastic waves trapped in the separating bodies, etc.

1.2 Classification of Methods for Analyzing Impact

In order to classify collisions into specific types which require distinct methods of analysis, we need to think about the deformations that develop during collision, the distribution of these deformations in each of the colliding bodies, and *how these deformations affect the period of contact*. In general there are four types of analysis for low speed collisions, associated with particle impact, rigid body impact, transverse impact on flexible bodies (i.e. transverse wave propagation or vibrations) and axial impact

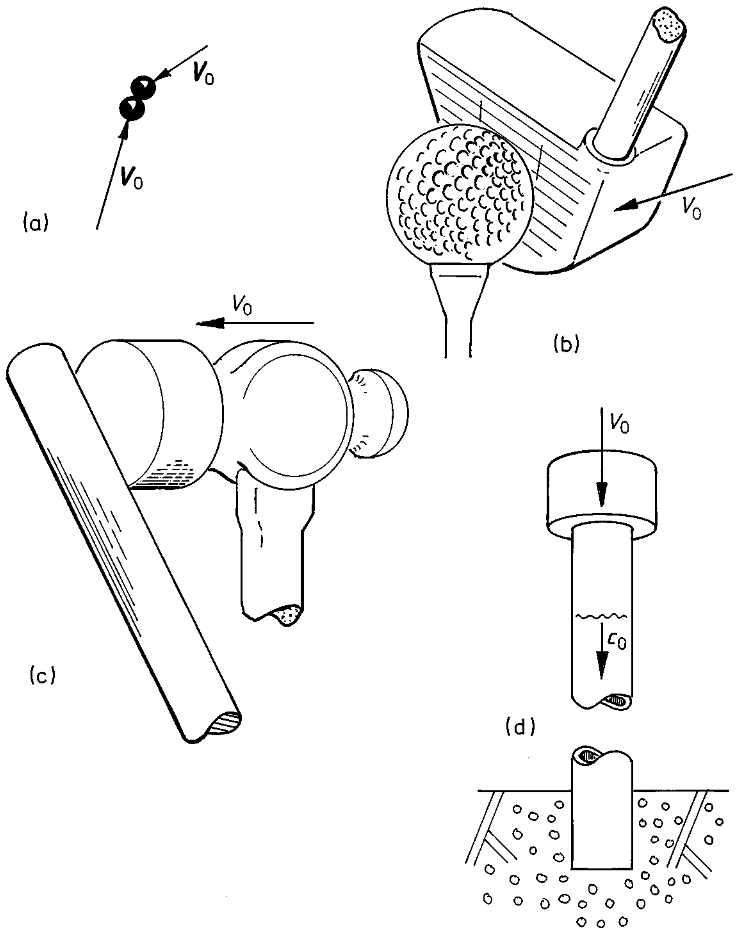


Figure 1.2. Impact problems requiring different analytical approaches: (a) particle impact (stereo-mechanical), (b) rigid body impact, (c) transverse deformations of flexible bodies and (d) axial deformation of flexible bodies.

on flexible bodies (i.e. longitudinal wave propagation). A typical example where each method applies is illustrated in Fig. 1.2.

(a) ***Particle impact*** is an analytical approximation that considers a normal component of interaction impulse only. By definition, particles are smooth and spherical. The source of the interaction force is unspecified, but presumably it is strong and the force has a very short range, so that the period of interaction is a negligibly small instant of time.

(b) ***Rigid body impact*** occurs between compact bodies where the contact area remains small in comparison with all section dimensions. Stresses generated in the contact area decrease rapidly with increasing radial distance from the contact region, so the internal energy of deformation is concentrated in a small region surrounding the interface. This small deforming region has large stiffness and acts much like a short but very stiff spring separating the colliding bodies at the contact point. The period of contact depends on the normal compliance of the contact region and an effective mass of the colliding bodies.

(c) ***Transverse impact on flexible bodies*** occurs when at least one of the bodies suffers bending as a result of the interface pressures in the contact area; bending is significant at points far from the contact area if the depth of the body in the direction normal to the common tangent plane is small in comparison with dimensions parallel to this plane. This bending reduces the interface pressure and prolongs the period of contact. Bending is a source of energy dissipation during collision in addition to the energy loss due to local deformation that arises from the vicinity of contact.

(d) ***Axial impact on flexible bodies*** generates longitudinal waves which affect the dynamic analysis of the bodies only if there is a boundary at some distance from the impact point which reflects the radiating wave back to the source; it reflects the outgoing wave as a coherent stress pulse that travels back to its source essentially undiminished in amplitude. In this case the time of contact for an impact depends on the transit time for a wave travelling between the impact surface and the distal surface. This time can be less than that for rigid body impact between hard bodies with convex surfaces.

1.2.1 Description of Rigid Body Impact

For bodies that are hard (i.e. with small compliance), only very small deformations are required to generate very large contact pressures; if the surfaces are initially nonconforming, the small deformations imply that the contact area remains small throughout the contact period. The interface pressure in this small contact area causes the initially nonconforming contact surfaces to deform until they conform or touch at most if not all points in a small contact area. Although the contact area remains small in comparison with cross-sectional dimensions of either body, the contact pressure is large, and it gives a large stress resultant, or *contact force*. The contact force is large enough to rapidly change the normal component of relative velocity across the small deforming region that surrounds the contact patch. The large contact force rapidly accelerates the bodies and thereby limits interference which would otherwise develop after incidence if the bodies did not deform.

Hence in a small region surrounding the contact area the colliding bodies are subjected to large stresses and corresponding strains that can exceed the yield strain of the material. At quite modest impact velocities (of the order of 0.1 m s^{-1} for structural metals) irreversible plastic deformation begins to dissipate some energy during the collision; consequently there is some loss of kinetic energy of relative motion in all but the most benign collisions. Although the stresses are large in the contact region, they decay rapidly with increasing distance from the contact surface. In an elastic body with a spherical coordinate system centered at the initial contact point, the radial component of stress, σ_r, decreases very rapidly with increasing radial distance r from the contact region (in an elastic solid σ_r decreases as r^{-2} in a 3D deformation field). For a hard body the corresponding rapid decrease in strain means that significant deformations occur only in a small region around the point of initial contact; consequently the deflection or indentation of the contact area remains very small.

Since the region of significant strain is not very deep or extensive, hard bodies have very small compliance (i.e., a large force generates only a small deflection). The small region of significant deformation is like a short stiff spring which is compressed between the two bodies during the period of contact. This spring has a large spring constant and gives a very brief period of contact. For example, a hard-thrown baseball or cricket ball striking a bat is in contact for a period of roughly 2 ms, while a steel hammer striking a nail is in contact for a period of about 0.2 ms. The contact duration for the hammer and nail is less because these colliding bodies are composed of harder materials than the ball and bat. Both collisions generate a maximum force on the order of 10 kN (i.e. roughly one ton).

From an analytical point of view, the most important consequence of the small compliance of hard bodies is that very little movement occurs during the very brief period of contact; i.e., despite large contact forces, there is insufficient time for the bodies to displace significantly during impact. This observation forms a fundamental hypothesis of *rigid body impact theory*, namely, that for hard bodies, analyses of impact can consider the period of contact to be vanishingly small. Consequently any changes in velocity occur instantaneously (i.e. in the initial or incident configuration). The system configuration at incidence is termed the *impact configuration*. This theory assumes there is no movement during the contact period.

Underlying Premises of Rigid Body Impact Theory

(a) In each of the colliding bodies the contact area remains small in comparison with both the cross-sectional dimensions and the depth of the body in the normal direction.

(b) The contact period is sufficiently brief that during contact the displacements are negligible and hence there are no changes in the system configuration; i.e., the contact period can be considered to be instantaneous.

If these conditions are approximately satisfied, rigid body impact theory can be applicable. In general this requires that the bodies are hard and that they suffer only small local deformation in collision. For a solid composed of material that is rate-independent, a small contact area results in significant strains only in a small region around the initial contact point. If the body is hard, the very limited region of significant deformations causes the compliance to be small and consequently the contact period to be very brief. This results in two major simplifications:

(a) Equations of planar motion are trivially integrable to obtain algebraic relations between velocity changes and the reaction impulse.[1]

(b) Finite active forces (e.g. gravitational or magnetic attraction) which act during the period of contact can be considered to be negligible, since these forces do no work during the collision.

During the contact period the only significant active forces are reactions at points of contact with other bodies; these reactions are induced by displacement constraints.

Figure 1.3 shows a collision where application of rigid body impact theory is appropriate. This series of high speed photographs shows development of a small area of contact when an initially stationary field hockey ball is struck by a hockey stick at an incident

[1] Because velocity changes can be obtained from algebraic relations, rigid body impact was one of the most important topics in dynamics before the development of calculus in the late seventeenth century.

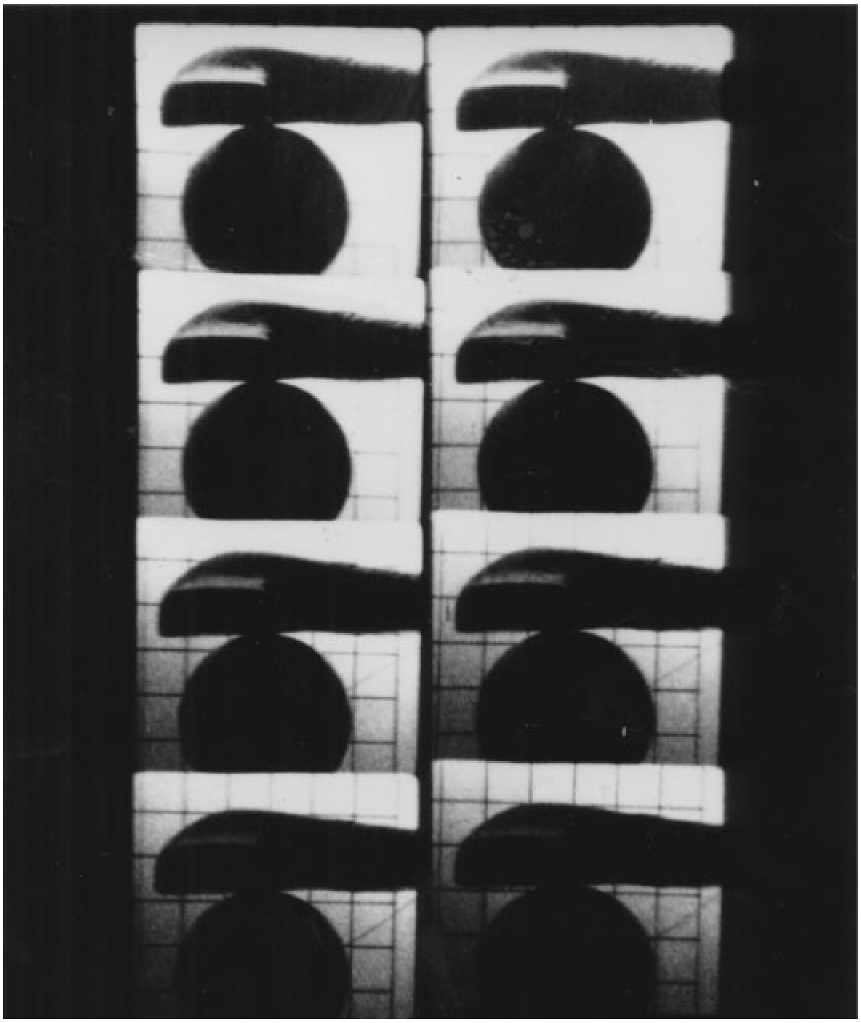


Figure 1.3. High speed photographs of hockey stick striking at 18 m s^{-1} (40 mph) against a stationary field hockey ball (dia. $D' = 74$ mm, mass $M' =$ 130 g). Interframe period $\tau = 0.0002$ s, contact duration $t_f \approx 0.0015$ s, and maximum normal force $F_c \approx 3900$ N.

speed of 18 m s^{-1}. During collision the contact area increases to a maximum radius a_c that remains small in comparison with the ball radius R'; in Fig. 1.3, $a_c/R' < 0.3$. The relatively small contact area is a consequence of the small normal compliance (or large elastic modulus) of both colliding bodies and the initial lack of conformation of the surfaces near the point of first contact.

A useful means of postulating rigid body impact theory is to suppose that two colliding bodies are separated by an infinitesimal deformable particle.[2] The deformable particle is

[2] The physical construct of a deformable particle separating contact points on colliding rigid bodies is mathematically equivalent to Keller's (1986) asymptotic method of integrating with respect to time the equations for relative acceleration of deformable bodies and then taking the limit as compliance (or contact period) becomes vanishingly small.

located between the point of initial contact on one body and that on the other, although these points are coincident. The physical construct of an infinitesimal compliant element separating two bodies at a point of contact allows variations in velocity during impact to be resolved as a function of the normal component of impulse. This normal component of impulse is equivalent to the integral of the normal contact force over the period of time after incidence. Since collisions between bodies with nonadhesive contact surfaces involve only compression of the deformable particle – never extension – the normal component of impulse is a monotonously increasing function of time after incidence. Thus variations in velocity during an instantaneous collision are resolved by choosing as an independent variable the normal component of impulse rather than time. This gives velocity changes which are a continuous (smooth) function of impulse.

There are three notable classes of impact problems where rigid body impact theory is not applicable if the impact parameters representing energy dissipation are to have any range of applicability.

(a) The first involves impulsive couples applied at the contact point. Since the contact area between rigid bodies is negligibly small, impulsive couples are inconsistent with rigid body impact theory. To relate a couple acting during impulse to physical processes, one must consider the distribution of deformation in the contact region. Then the couple due to a distribution of tangential force can be obtained from the law of friction and the first moment of tractions in a finite contact area about the common normal through the contact point.

(b) A second class of problems where rigid body impact theory does not apply is axial impact of collinear rods with plane ends. These are problems of one dimensional wave propagation where the contact area and cross-sectional area are equal because the contacting surfaces are conforming; in this case the contact area may not be small. For problems of wave propagation deformations and particle velocities far from the contact region are not insignificant. As a consequence, for one dimensional waves in long bars, the contact period is dependent on material properties and depth of the bars in a direction normal to the contact plane rather than on the compliance of local deformation near a point of initial contact.

(c) The third class of problems where rigid body theory is insufficient are transverse impacts on beams or plates where vibration energy is significant.

Collisions with Compliant Contact Region Between Otherwise Rigid Bodies

While most of our attention will be directed towards rigid body impact, there are cases where distribution of stress is significant in the region surrounding the contact area. These problems require consideration of details of local deformation of the colliding bodies near the point of initial contact; they are analyzed in Chapters 6 and 8. The most important example may be collisions against multibody systems where the contact points between bodies transmit the action from one body to the next; in general, this case requires consideration of the compliance at each contact. Considerations of local compliance may be represented by discrete elements such as springs and dashpots or they can be obtained from continuum theory.

For collisions between systems of hard bodies, it is necessary to consider *local displacement* in each contact region although *global displacements* are negligibly small; i.e. different scales of displacement are significant for different analytical purposes. The relatively small displacements that generate large contact forces are required to analyze interactions between spatially discrete points of contact. If the bodies are hard however, these same displacements may be sufficiently small so that they have negligible effect on the inertia properties; i.e. during collision any changes in the inertia properties are insignificant despite the small local deformations.

1.2.2 Description of Transverse Impact on Flexible Bodies

Transverse impact on plates, shells or slender bars results in significant flexural deformations of the colliding members both during and following the contact period. In these cases the stiffness of the contact region depends on flexural rigidity of the bodies in addition to continuum properties of the region immediately adjacent to the contact area; i.e., it is no longer sufficient to suppose that a small deforming region is surrounded by a rigid body. Rather, flexural rigidity is usually the more important factor for contact stiffness when impact occurs on a surface of a plate or shell structural component.

1.2.3 Description of Axial Impact on Flexible Bodies

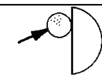
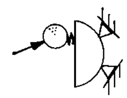
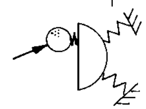
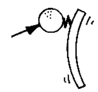
Elastic or elastic–plastic waves radiating from the impact site are present in every impact between deformable bodies – in a deformable body it is these radiating waves that transmit variations in velocity and stress from the contact region to the remainder of the body. Waves are an important consideration for obtaining a description of the dynamic response of the bodies, however, only if the period of collision is determined by wave effects. This is the case for axial impact acting uniformly over one end of a slender bar if the far, or *distal*, end of the bar imposes a reflective boundary condition. Similarly, for radial impact at the tip of a cone, elastic waves are important if the cone is truncated by a spherical surface with a center of curvature at the apex. In these cases where the impact point is also a focal point for some reflective distal surface, the wave radiating from the impact point is reflected from the distal surface and then travels back to the source, where it affects the contact pressure. On the other hand, if different parts of the outgoing stress wave encounter boundaries at various times and the surfaces are not normal to the direction of propagation, the wave will be reflected in directions that are not towards the impact point; while the outgoing wave changes the momentum of the body, this wave is diffused rather than returning to the source as a coherent wave that can change the contact pressure and thereby affect the contact duration.

1.2.4 Applicability of Theories for Low Speed Impact

This text presents several different methods for analyzing changes in velocity (and contact forces) resulting from low speed impact, i.e. impact slow enough that the bodies are deformed imperceptibly only. These theories are listed in Table 1.1 with descriptions of the differences and an indication of the range of applicability for each.

The stereomechanical theory is a relationship between incident and final conditions; it results in discontinuous changes in velocity at impact. In this book a more sophisticated *rigid body* theory is developed – a theory in which the changes in velocity are a continuous function of the normal component of the impulse p at the contact point. This theory

Table 1.1. *Applicability of Theories for Oblique, Low Speed Impact*

Impact Theory	Independent Variable	Coeff. of Restitution[a]	Angle of Incidence at Impact Point,[b] ψ_0	Spatial Gradient of Contact Compliance,[c] χ^{-1}	(Impact Point Compliance)/(Structural Compliance)[d]	Computational Effort	Illustration
Stereomechanical[e]	None	e, e_0	$> \tan^{-1}(\mu\beta_1/\beta_3)$	>1	$\gg 1$	Low	
Rigid body[f]	Impulse p	e_*	$> \tan^{-1}(\mu\beta_1/\beta_3)^g$	>1 (sequential) $\ll 1$(simultaneous)	$\gg 1$	Low	
Compliant contact[f]	Time t	e_*	All	All	$\gg 1$	Moderate	
Continuum[f]	Time t	None	All	All	All	High	

[a] e, e_0, e_* = kinematic, kinetic, energetic coefficients of restitution.
[b] μ = Amontons–Coulomb coefficient of limiting friction; β_1, β_3 = inertia coefficients.
[c] Distributed points of contact.
[d] Flexible bodies.
[e] Nonsmooth dynamics.
[f] Smooth dynamics.
[g] Or negligible tangential compliance.

results from considering that the coincident points of contact on two colliding bodies are separated by an infinitesimal deformable particle – a particle that represents local deformation around the small area of contact. With this artifice, the analysis can follow the process of slip and/or slip–stick between coincident contact points if the contact region has negligible tangential compliance. Rigid body theories are useful for analyzing two body impact between compact bodies composed of stiff materials; however, they have limited applicability for multibody impact problems.

When applied to multibody problems, rigid body theories can give accurate results only if the normal compliance of the point of external impact is very small or large in comparison with the compliance of any connections with adjacent bodies. If compliance of the point of external impact is much smaller than that of all connections to adjacent bodies, at the connections the maximum reaction force occurs well after the termination of contact at the external impact point, so that the reactions essentially occur sequentially. Small impact compliance results in a wave of reaction that travels away from the point of external impact at a speed that depends on the inertia of the system and the local compliance at each connecting joint or contact point. On the other hand, if the normal compliance of the point of external impact is very large in comparison with compliance of any connections to adjacent bodies, the reactions at the connections occur simultaneously with the external impact force. Only in these limiting cases can the dynamic interaction between connected bodies be accurately represented with an assumption of either sequential or simultaneous reactions. Generally the reaction forces at points of contact arise from infinitesimal relative displacements that develop during impact; these reaction forces are coupled, since sometimes they overlap.

If however other points of contact or cross-sections of the body have compliance of the same order of magnitude as that at any point of external impact, then the effect of these flexibilities must be incorporated into the dynamic model of the system. If the compliant elements are local to joints or other small regions of the system, an analytical model with local compliance may be satisfactory; e.g. see Chapter 8. On the other hand, if the body is slender, so that significant structural deformations develop during impact, either a wave propagation or a structural vibration analysis may be required; see Chapter 7 or 9. Whether the distributed compliance is local to joints or continuously distributed throughout a flexible structure, these theories require a time-dependent analysis to obtain reaction forces that develop during contact and the changes in velocity caused by these forces.

Hence the selection of an appropriate theory depends on structural details and the degree of refinement required to obtain the desired information.

1.3 Principles of Dynamics

1.3.1 Particle Kinetics

The fundamental form of most principles of dynamics is in terms of the dynamics of a particle. A *particle* is a body of negligible or infinitesimal size, i.e. a point mass. The particle is the building block that will be used to develop the dynamics of impact for either rigid or deformable solids. A particle of mass M moving with velocity $\mathbf{V}$ has *momentum* $M\mathbf{V}$. If a resultant force $\mathbf{F}$ acts on the particle, this causes a change in momentum in accord with Newton's second law of motion.

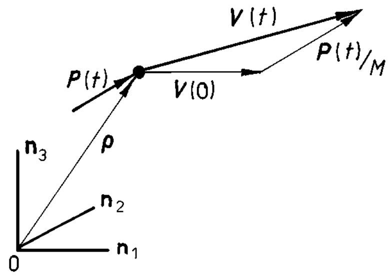


Figure 1.4. Change in velocity of particle with mass M resulting from impulse $\mathbf{P}(t)$.

Law II: The momentum $M\mathbf{V}$ of a particle has a rate of change with respect to time that is proportional to and in the direction of any resultant force $\mathbf{F}(t)$ acting on the particle[3]:

$$d(M\mathbf{V})/dt = \mathbf{F} \tag{1.1}$$

Usually the particle mass is constant, so that Eq. (1.1) can be integrated to obtain the changes in velocity as a continuous function of the *impulse* $\mathbf{P}(t)$:

$$\mathbf{V}(t) - \mathbf{V}(0) = M^{-1} \int_0^t \mathbf{F}(t')\,dt' \equiv M^{-1}\mathbf{P}(t) \tag{1.2}$$

This vector expression is illustrated in Fig. 1.4.

The interaction of two particles B and B′ that collide at time $t = 0$ generates active forces $\mathbf{F}(t)$ and $\mathbf{F}'(t)$ that act on each particle respectively, during the period of interaction, $0 < t < t_f$ – these forces of interaction act to prevent interpenetration. The particular nature of interaction forces depends on their source: whether they are due to contact forces between solid bodies that cannot interpenetrate, or are interatomic forces acting between atomic particles. In any case the force on each particle acts solely in the radial direction. These interaction forces are related by Newton's third law of motion.

Law III: Two interacting bodies have forces of action and reaction that are equal in magnitude, opposite in direction and collinear:

$$\mathbf{F}' = -\mathbf{F} \tag{1.3}$$

Laws II and III are the basis for impulse–momentum methods of analyzing impact. Let particle B have mass M, and particle B′ have mass M'. Integration of (1.3) gives equal but opposite impulses $-\mathbf{P}'(t) = \mathbf{P}(t)$, so that equations of motion for the *relative velocity* $\mathbf{v} \equiv \mathbf{V} - \mathbf{V}'$ can be obtained as

$$\mathbf{v}(t) = \mathbf{v}(0) + m^{-1}\mathbf{P}(t), \qquad m^{-1} = M^{-1} + M'^{-1} \tag{1.4}$$

[3] Newton's second law is valid only in an inertial reference frame or a frame translating at constant speed relative to an inertial reference frame. In practice a reference frame is usually considered to be fixed relative to a body, such as the earth, which may be moving. Whether or not such a reference frame can be considered to be inertial depends on the magnitude of the acceleration being calculated in comparison with the acceleration of the reference body, i.e. whether or not the acceleration of the reference frame is negligible.

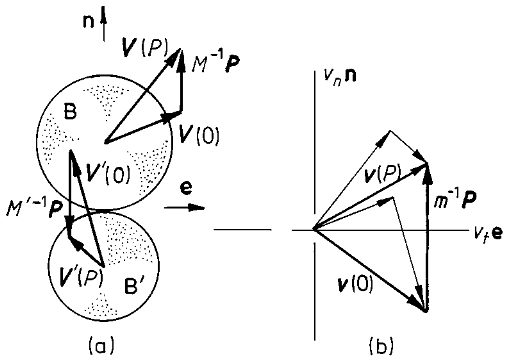


Figure 1.5. (a) Equal but opposite normal impulses **P** on a pair of colliding bodies with masses M and M' result in velocity changes $M^{-1}\mathbf{P}$ and $-M'^{-1}\mathbf{P}$ respectively. (b) The light lines are the initial and the final velocity for each body, while the heavy lines are the initial relative velocity $\mathbf{v}(0)$, the final relative velocity $\mathbf{v}(P)$ and the change $m^{-1}\mathbf{P}$ in relative velocity.

where m is the *effective mass*. The change of variables from velocity $\mathbf{V}(t)$ in an inertial reference frame to relative velocity $\mathbf{v}(t)$ is illustrated in Fig. 1.5. Equation (1.4) is an equation of relative motion that is applicable in the limit as the period of contact approaches zero ($t_f \to 0$); this equation is the basis of smooth dynamics of collision for particles and rigid bodies.

Example 1.1 A golf ball has mass $M = 61$ g. When hit by a heavy club the ball acquires a speed of 44.6 m s^{-1} (100 mph) during a contact duration $t_f = 0.4$ ms. Assume that the force–deflection relation is linear, and calculate an estimate of the maximum force $F_{\max}$ acting on the ball.

Solution

Effective mass $m = 0.061$ kg.
Initial relative velocity $v(0) = v_0 = -44.6$ m s^{-1}.

(a) Linear spring $\Rightarrow$ simple harmonic motion for relative displacement δ at frequency ω where $\omega t_f = \pi$.
(b) Change in momentum of relative motion = impulse, Eq. (1.4):

$$mv_0 = \int_0^{t_f} F(t)\,dt = \int_0^{t_f} F_{\max}\sin(\omega t)\,dt$$

$$\Rightarrow \quad F_{\max} = 21.4\ \text{kN}\,(\approx 2\ \text{tons})$$

1.3.2 Kinetics for a Set of Particles

For a set of n particles where the ith particle has mass M_i and velocity $\mathbf{V}_i$ the equations of translational motion can be expressed as

$$\frac{d}{dt}\sum_{i=1}^{n} M_i \mathbf{V}_i = \sum_{i=1}^{n} \mathbf{F}_i + \sum_{i=1}^{n}\sum_{k=1}^{n} \mathbf{F}'_{ik}, \qquad k \neq i$$