

Contents

<i>Preface</i>	<i>page</i>	xiv
1 CATEGORIES		1
1.1 FUNDAMENTAL PROPERTIES OF CATEGORIES		1
1.1.1 The definition		2
1.1.2 Some examples		3
1.1.3 The axioms		5
1.1.4 Chirality		5
1.1.5 The mirror		6
1.1.6 The opposite category		8
1.1.7 The principle of duality		8
1.1.8 Subcategories		9
1.1.9 Full subcategories		10
1.1.10 Some remarks on set theory and small categories		10
1.1.11 Product categories		11
1.1.13 Infinite products		13
1.1.14 Morphism categories		13
Exercises		14
1.2 FUNCTORS		16
1.2.1 Functors		17
1.2.2 Some examples		18
1.2.3 Free modules		19
1.2.4 The product		19
1.2.5 Restriction		20
1.2.6 Appearances and chirality		20
1.2.7 The morphism functors		22
1.2.8 The category $\mathcal{C}_{\mathcal{A}\mathcal{T}}$		23
1.2.9 Fibre categories		23
1.2.10 Examples		24
		vii

viii	<i>Contents</i>
Exercises	25
1.3 NATURAL TRANSFORMATIONS	26
1.3.1 Natural transformations	26
1.3.2 Examples	27
1.3.3 Natural isomorphisms	30
1.3.4 Matrices and bases of free modules: a summary	30
1.3.5 Example: matrices and bases of free modules	32
1.3.6 Multifunctors	33
1.3.7 Adjoint functors	35
1.3.8 Example: free modules	36
1.3.9 Functor categories	36
1.3.10 Diagrams	37
1.3.11 Equivalence of categories	38
1.3.12 Example: standard bases	39
1.3.13 Faithful, full and dense	39
1.3.15 Skeletons	39
Exercises	41
1.4 UNIVERSAL OBJECTS	47
1.4.1 Initial objects	48
1.4.2 Universal objects	49
1.4.3 Free modules revisited	49
1.4.8 Universal constructions	55
1.4.9 Terminal objects	56
1.4.10 The direct sum revisited	56
1.4.11 The product	57
1.4.12 The coproduct	58
1.4.13 Arbitrary products and coproducts	58
1.4.14 Zero objects	59
1.4.15 Kernels and cokernels	59
1.4.16 Universal properties	60
1.4.17 Pleasure versus guilt: the universal dilemma	61
1.4.18 Kernels of natural transformations	61
1.4.19 Some history	62
Exercises	63
2 CATEGORIES AND EXACT SEQUENCES	68
2.1 THE HOMOMORPHISM FUNCTORS	68
2.1.1 Basic properties	69
2.1.2 Exact sequences	70
2.1.5 Short exact sequences	73
2.1.7 Projective and injective modules	75

<i>Contents</i>	ix
2.1.9 Homomorphism functors arising from bimodules	76
2.1.10 The extension functors	77
Exercises	78
2.2 ADDITIVE CATEGORIES	83
2.2.1 Preadditive categories	84
2.2.2 Preadditive subcategories	85
2.2.3 Monomorphisms and epimorphisms	85
2.2.5 Example: the opposite category	87
2.2.6 Example: topological abelian groups	87
2.2.7 Kernel and cokernel	88
2.2.9 Short exact sequences	89
2.2.10 Projective and injective objects	89
2.2.11 Additive categories	91
2.2.12 Additive subcategories	92
2.2.13 Examples	92
2.2.16 Morphism categories	93
2.2.18 Additive functors	94
2.2.21 Functor categories	95
Exercises	97
2.3 ABELIAN CATEGORIES	99
2.3.1 The definition	99
2.3.3 Module-like behaviour	101
2.3.4 Example	101
2.3.7 More examples	102
2.3.10 Product and morphism categories	104
2.3.13 Module categories	105
2.3.14 Functor categories	106
2.3.16 Direct sums of categories	107
2.3.19 Infinite direct sums of categories	108
2.3.20 Dedekind domains: a review	109
2.3.21 Module categories over Dedekind domains	112
2.3.22 The Embedding Theorems	113
2.3.23 Example: The Famous Five Lemma	113
Exercises	114
2.4 EXACT CATEGORIES	118
2.4.1 G -exact categories	119
2.4.2 Split and repletely G -exact categories	120
2.4.4 Relative exact categories	121
2.4.5 On terminology	121
2.4.6 Exact functors	122

x	<i>Contents</i>
2.4.7 Examples	122
2.4.9 The Grothendieck group	123
2.4.10 Q -exact categories	124
2.4.11 Comments on the axioms	126
2.4.13 Exact subcategories	127
Exercises	128
3 CHANGE OF RINGS	135
3.1 THE TENSOR PRODUCT	135
3.1.1 The definition	136
3.1.2 The construction	137
3.1.3 Bimodule structures	138
3.1.6 Functorial properties of tensor products	139
3.1.9 Fixing the first argument	141
3.1.16 Return of the dyads	145
3.1.18 The adjointness of the functors Hom and \otimes	146
3.1.20 An equivalence of categories	148
Exercises	149
3.2 EXACTNESS OF THE TENSOR PRODUCT	150
3.2.1 Flat modules	151
3.2.6 The functors Tor_n^R	153
3.2.7 Criteria for flatness and Villamayor's Lemma	154
3.2.14 A pairing on $\mathcal{M}OD_R$	160
Exercises	161
3.3 CHANGE OF SCALARS	163
3.3.1 Restriction	164
3.3.4 Extension	166
3.3.8 Exactness	168
3.3.9 An identification	169
3.3.10 The quotient functor	169
3.3.16 \mathbb{R} and \mathbb{C}	172
3.3.18 Skew fields unbalanced	173
3.3.21 The definitions for left modules	175
3.3.22 The twisting of modules	175
3.3.26 Group rings	178
Exercises	179
4 THE MORITA THEORY	184
4.1 PROJECTIVE GENERATORS	184
4.1.1 The dual of a module	185
4.1.3 The dual of a free module	186
4.1.4 Endomorphisms of a free left module	187

<i>Contents</i>	xi
4.1.5 The evaluation homomorphisms	188
4.1.6 Projective modules	189
4.1.8 Some identifications	190
4.1.10 Generators	191
4.1.12 Progenerators	192
4.1.16 Commutative domains	196
Exercises	198
4.2 MORITA EQUIVALENCE	204
4.2.1 The definition and first results	205
4.2.4 Further developments	207
4.2.5 Properties preserved by Morita equivalence	207
4.2.12 An illustration: matrix rings	210
4.2.15 An illustration: orders over Dedekind domains	211
4.2.16 The Picard Group	213
4.2.18 Definition of $\text{Pic}(R)$	214
4.2.22 Orders	216
Exercises	217
5 LIMITS IN CATEGORIES	222
5.1 DIRECT LIMITS	223
5.1.1 Directed sets	223
5.1.2 Examples	224
5.1.3 Direct systems	225
5.1.4 Construction of the direct limit	226
5.1.6 Some examples	227
5.1.7 Basic properties of direct limits	229
5.1.14 Quasicyclic groups	231
5.1.15 Cofinality	231
5.1.16 Generalizations	232
5.1.17 Notation for left modules	233
5.1.20 Matrices again	235
Exercises	235
5.2 DIRECT LIMITS, FLAT MODULES AND RINGS	238
5.2.1 Construction of flat modules	238
5.2.8 Direct limits of rings	241
5.2.9 Examples: yet more matrices	242
5.2.10 Von Neumann regular rings	243
5.2.15 An example: idempotents all decompose	244
Exercises	245
5.3 INVERSE LIMITS	246
5.3.1 The definition	246

xii	<i>Contents</i>
5.3.3	The p -adic integers 248
5.3.4	Sums and coproducts as limits 248
	Exercises 249
6	LOCALIZATION 252
6.1	LOCALIZATION FOR RINGS 252
6.1.1	Ore sets 253
6.1.3	Examples 254
6.1.4	Basic properties of rings of fractions 254
6.1.5	The construction of the ring of fractions 255
6.1.7	Definition of the ring of fractions 256
6.1.10	R_{Σ} is a ring 258
6.1.13	Local rings 260
6.1.17	The maximal ring of fractions 261
	Exercises 262
6.2	LOCALIZATION FOR MODULES 263
6.2.1	Localization and torsion 264
6.2.5	Some categories 267
6.2.8	Modules over the ring of fractions 268
6.2.16	Ore domains 271
6.2.20	Left–right symmetry 272
6.2.21	An asymmetric example 272
6.2.22	More symmetry 273
	Exercises 273
6.3	CATEGORICAL LOCALIZATION 276
6.3.1	Serre subcategories 277
6.3.2	Examples 277
6.3.3	\mathcal{C} -isomorphisms 277
6.3.4	Denominator sets 278
6.3.5	Some direct systems of morphisms 280
6.3.6	The quotient category 281
6.3.7	The quotient functor 282
6.3.8	The universal property 283
6.3.9	Some comments on localization in general 283
	Exercises 285
7	LOCAL-GLOBAL METHODS 289
7.1	THE COMPLETION OF A DEDEKIND DOMAIN 290
7.1.1	Valuations 290
7.1.4	Valuations and Dedekind domains 292
7.1.7	Generalizations 293
7.1.8	Cauchy sequences 293

<i>Contents</i>	xiii
7.1.10 Completeness	294
7.1.12 Constructing the completion	295
7.1.13 Extending the valuation	296
7.1.19 Power series	301
7.1.20 Modules over complete rings	302
7.1.22 Completion of modules	303
7.1.25 Adèles	305
7.1.30 Completions in general	308
Exercises	308
7.2 THE PROJECTIVE MODULE LIFTING PROBLEM	311
7.2.1 Some illustrations	311
7.2.2 Idempotents	312
7.2.3 Semiperfect rings	312
7.2.6 Orders over complete valuation rings	313
7.2.10 Projective lifting	314
7.2.13 Functoriality of lifting	316
7.2.14 Indecomposable modules	317
Exercises	318
7.3 LOCAL-GLOBAL METHODS FOR ORDERS	319
7.3.1 Lattices	320
7.3.2 Full lattices	320
7.3.5 An anonymous invariant	321
7.3.10 Completions of lattices	325
7.3.15 Adèles for modules	328
7.3.18 Adèles for spaces	330
7.3.20 Lattices over orders	331
7.3.22 Projective modules	332
7.3.30 Maximal and hereditary orders	338
7.3.31 The conductor	338
7.3.32 Concluding remarks	339
Exercises	340
<i>References</i>	344
<i>Index</i>	350