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CATEGORIES

Category theory provides the language for the discussions in this text and it is also an inescapable foundation for any treatment of K-theory. In this chapter, we set out the fundamental ideas of category theory, and we give a first application of the power of categorical methods. These ideas are introduced in three stages, each of which occupies one section. To start, we define categories themselves. Next, we consider functors, which are the tools for moving between different categories, and then natural transformations, which allow us to compare different functors. In the final, fourth, section of the chapter, we apply these basic notions to investigate universal constructions and universal objects. The idea of 'universality' reveals the common properties of apparently diverse objects, such as free modules and free groups, and it provides the framework for many of the definitions and constructions that we make later in this text.

1.1 FUNDAMENTAL PROPERTIES OF CATEGORIES

We commence our discussion of the theory of categories with the axiomatic definition of a category, together with a selection of techniques for the manufacture of new categories from old.

Because our main interest in this text is the application of category theory to module theory, many of our illustrations are obtained by considering various types of module. We shall also see some categories based on other algebraic entities such as sets and groups.

The principal innovation in our discussion stems from our need to use both right modules and left modules. Our view is that it is best to write operators opposite scalars as far as is practicable, so that a homomorphism between right modules is to be a left operator, while a homomorphism of left modules is to be a right operator. We also wish to compose homomorphisms in the natural way, by 'associativity' – more formal definitions are given below in (1.1.3) and (1.1.4). The consequence is that we obtain two versions of the axiom for the composition of morphisms in an abstract category. Thus we arrive at two kinds of abstract category, a 'right' category and a 'left' category, which are modeled on the corresponding categories of modules. We call this phenomenon *chirality*. It is worth remarking that the distinction between left and right modules was made at a very early stage in the modern development of module theory [Noether & Schmeidler 1920].

The distinction between right and left categories does not seem to have been made explicit before now, and it could be avoided by using a technique given in (1.1.5). However, it seems to us to be more natural to allow both notations for categories, rather than suppressing one notation artificially.

We also discuss some points from set theory which arise from future applications in K-theory, where one needs to be able to work with 'small' categories, that is, categories whose objects can all be taken to be members of some set, which may itself be very big. In general, the objects of a category need not be contained in a set. We are particularly indebted to Wilfrid Hodges for his helpful comments on these questions.

1.1.1 The definition

Informally, a category consists of a collection of mathematical entities, such as the right R-modules over a given ring R, which can be recognised as sharing a common structure, together with a collection of mappings between these entities that respect this structure; for R-modules, we would expect these to be the R-module homomorphisms.

The entities which share the common structure are known as the 'objects' of the category, while the structure-preserving maps are the 'morphisms' of the category. Thus, to define a *category* $\mathcal{C}$ in general, we must specify the following data.

Cat 1. A class $\mathrm{Ob}\,\mathcal{C}$. Members of $\mathrm{Ob}\,\mathcal{C}$ are called the *objects* of $\mathcal{C}$.

Cat 2. For each ordered pair C, D of objects of $\mathcal{C}$, there is a set $\mathrm{Mor}_{\mathcal{C}}(C, D)$; the elements of $\mathrm{Mor}_{\mathcal{C}}(C, D)$ are called the *morphisms* from C to D in $\mathcal{C}$.

It may happen that $\mathrm{Mor}_{\mathcal{C}}(C, D)$ is the empty set.

Given $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, the object C is called the *domain* of α and D the *codomain*. An arrow

$$\alpha : C \longrightarrow D$$

is often used to indicate that α is a morphism from C to D.

These objects and morphisms obey some axioms, but before listing these we note two deeply entrenched conventions and give some examples.

The first convention is to write $C \in \mathcal{C}$ rather than $C \in \mathrm{Ob}\,\mathcal{C}$ when C is an object of $\mathcal{C}$. This reflects the fact that a category is usually considered to be defined in terms of the class $\mathrm{Ob}\,\mathcal{C}$. For this reason, a category is often named after its objects.

Second, the term 'morphism' is used mostly when we are considering categories in the abstract; in concrete situations, the morphisms are often the homomorphisms between some familiar objects, such as modules, groups or rings. In such a case, one writes $\mathrm{Hom}(C, D)$ rather than $\mathrm{Mor}_{\mathcal{C}}(C, D)$.

1.1.2 Some examples

$\mathcal{S}\mathrm{ET}$ The class $\mathrm{Ob}\,\mathcal{S}\mathrm{ET}$ of objects is the class of all sets and the set of morphisms from X to Y is the set of all mappings from X to Y. (Some authors denote this category $\mathcal{E}\mathrm{NS}$).

We write $\mathrm{Map}(X, Y)$ rather than $\mathrm{Mor}(X, Y)$.

$\mathcal{G}\mathrm{P}$ The objects are the groups, with group homomorphisms as morphisms.

$\mathcal{R}\mathrm{ING}$ The objects are the rings (which we require to have an identity element), the morphisms from R to S being the ring homomorphisms, which, we insist, send the identity of R to the identity of S.

$\mathcal{R}\mathrm{NG}$ The objects are now the nonunital rings, which are rings except that they need not possess an identity element. A morphism from R to S is a homomorphism of nonunital rings, which is the same as a ring homomorphism except that there can be no requirement that the identity of R is sent to the identity of S, even when R and S are actually rings.

$\mathcal{M}\mathrm{OD}_R$ Given a ring R, we form the category of all right R-modules. Thus $\mathrm{Ob}(\mathcal{M}\mathrm{OD}_R)$ is the class of all right R-modules and the set of morphisms from M to N is the set of all R-module homomorphisms from M to N, which we write $\mathrm{Hom}(M, N)$, $\mathrm{Hom}_R(M, N)$ or $\mathrm{Hom}(M_R, N_R)$ according to context.

${}_R\mathcal{M}\mathrm{OD}$ Similarly, we form the category of left R-modules and homomorphisms of left R-modules.

$\mathcal{A}\mathrm{B}$ The category of abelian groups – the objects are abelian groups, the morphisms are group homomorphisms.

Since an abelian group has a uniquely defined structure as a right $\mathbb{Z}$-module ([BK: IRM] (1.2.2)), $\mathcal{A}\mathrm{B}$ is $\mathcal{M}\mathrm{OD}_{\mathbb{Z}}$ with another name. Despite the fact that any right $\mathbb{Z}$-module can be regarded equally as a left $\mathbb{Z}$-module, with $am = ma$ for any integer a and element m of M, it is not quite true that $\mathcal{A}\text{B}$ is the same as the category ${}_{\mathbb{Z}}\mathcal{M}\text{OD}$ of left $\mathbb{Z}$-modules. The reason for the distinction will be discussed further in (1.1.5).

$\mathcal{T}\text{OP}$ This important non-algebraic example of a category has topological spaces as objects and continuous maps as morphisms.

Λ It is sometimes convenient to regard a partially ordered set as a category. A *partially ordered* set is a set Λ together with an order relation $\leq$ on Λ which satisfies the following requirements.

PO 1. **Reflexivity:**
$\lambda \leq \lambda$ for each $\lambda \in \Lambda$.

PO 2. **Transitivity:**
if $\lambda \leq \mu$ and $\mu \leq \nu$ for λ, μ and ν in Λ, then $\lambda \leq \nu$ also.

A partially ordered set is said to be *proper* if the following axiom also holds.

PO 3. If $\lambda \leq \mu$ and $\mu \leq \lambda$ for λ and μ in Λ, then $\lambda = \mu$.

Here are two examples that will be generalized in (5.1).

(a) Let Λ be the set of nonzero ideals of $\mathbb{Z}$, ordered by reverse inclusion:

$$I \leq J \iff J \subseteq I.$$

Then Λ is a proper partially ordered set.

(b) Take Σ to be the set of nonzero elements of $\mathbb{Z}$, ordered by division:

$$a \leq b \iff ax = b \text{ for some } x \in \mathbb{Z}.$$

Then Σ is partially ordered, but not proper.

Given a partially ordered set Λ, we can view Λ as a category whose objects are the elements $\lambda, \mu, \ldots$ of Λ. If $\lambda \leq \mu$, then $\mathrm{Mor}(\lambda, \mu)$ contains a single element $\iota^{\lambda\mu}$, and $\mathrm{Mor}(\lambda, \mu)$ is empty otherwise.

$\mathcal{O}\text{RD}$ The class of all ordered sets can be considered to be a category. We say that a set Λ is *totally ordered*, or simply *ordered*, if it is a proper partially ordered set in which any two members are comparable, that is, the following axiom holds.

TO. If λ and μ are in Λ, then either $\lambda \leq \mu$ or $\mu \leq \lambda$.

Then the objects of $\mathcal{O}\text{RD}$ are the ordered sets Λ and a morphism $\alpha : \Lambda \to \Sigma$ of ordered sets is a mapping that preserves the order: if $\lambda \leq \mu$ in Λ, then $\alpha\lambda \leq \alpha\mu$ in Σ.

1.1.3 The axioms

Now that we have some examples in mind, we list the axioms that the morphisms of an abstract category $\mathcal{C}$ are required to satisfy. These correspond to the properties of the homomorphisms in the category $\mathcal{M}od_R$, provided that we write homomorphisms on the left and that we use the consequent natural convention for composing homomorphisms. Thus, if α is a right R-module homomorphism from M_R to N_R and m is in M, then αm denotes the image of m in N, and if β is a homomorphism from N_R to P_R, then

$$(\beta\alpha)m = \beta(\alpha m).$$

Cat 3. For each object C of $\mathcal{C}$, there is a distinguished morphism

$$id_C \in \mathrm{Mor}_{\mathcal{C}}(C, C),$$

called the *identity* morphism of C.

Cat 4. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$ and $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$, then there is a morphism

$$\beta\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, E),$$

called the *product* or *composite* of α and β; α and β are said to be *composable.*

Cat 5. For any morphism $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$,

$$(id_D)\alpha = \alpha = \alpha(id_C).$$

Cat 6. If $\alpha \in \mathrm{Mor}_{\mathcal{C}}(C, D)$, $\beta \in \mathrm{Mor}_{\mathcal{C}}(D, E)$ and $\gamma \in \mathrm{Mor}_{\mathcal{C}}(E, F)$, then

$$\gamma(\beta\alpha) = (\gamma\beta)\alpha.$$

With regard to Cat 4, note that the product of two morphisms α, β is defined precisely when the codomain of α is the same as the domain of β. Observe also that the manner of writing the product (as $\beta\alpha$ rather than $\alpha\beta$) forms a part of this axiom.

Statements such as Cat 6 are sometimes formulated as 'the product is associative when defined'.

1.1.4 Chirality

It seems to us that a category satisfying the axioms above should, strictly speaking, be called a *right* category, since the form of the axioms mimics the natural form of composition of homomorphisms of *right* R-modules when these homomorphisms are written on the left, as is our convention.

When we consider left modules, we prefer to put homomorphisms on the

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right. Thus, given a homomorphism of left R-modules $\alpha : {}_RM \to {}_RN$ and an element m in M, α sends m to $m\alpha$ rather than αm. If $\beta : {}_RN \to {}_RP$ is also a homomorphism, then the product $\alpha\beta$ is given by the rule

$$m(\alpha\beta) = (m\alpha)\beta.$$

With this convention, we obtain a modified list of axioms that defines a *left* category, as follows.

Cat 4^ℓ. If $\alpha \in \text{Mor}_\mathcal{C}(C, D)$ and $\beta \in \text{Mor}_\mathcal{C}(D, E)$, then there is a morphism

$$\alpha\beta \in \text{Mor}\,_\mathcal{C}(C, E).$$

Cat 5^ℓ. For any morphism $\alpha \in \text{Mor}_\mathcal{C}(C, D)$,

$$(id_C)\alpha = \alpha = \alpha(id_D).$$

Cat 6^ℓ. If $\alpha \in \text{Mor}_\mathcal{C}(C, D)$, $\beta \in \text{Mor}_\mathcal{C}(D, E)$ and $\gamma \in \text{Mor}_\mathcal{C}(E, F)$, then

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

The prime example of a left category is ${}_R\mathcal{M}\text{OD}$, the category of all left R-modules. Also, most group theorists in effect treat $\mathcal{G}\mathcal{P}$ as a left category.

As we noted in (1.1.2), a partially ordered set Λ can be viewed as a category. It is convenient to regard Λ as a left category, since the left-category composition reads

$$\iota^{\lambda\mu}\iota^{\mu\nu} = \iota^{\lambda\nu},$$

where $\iota^{\lambda\mu}$ is the unique morphism from λ to μ (when $\lambda \leq \mu$).

We introduce the term *chirality* to distinguish the two kinds of category, a left or right category having correspondingly left or right chirality.

1.1.5 The mirror

There is a purely formal method that allows us to switch between categories of opposite chiralities. Given a left category $\mathcal{C}$, we manufacture a right category $\mathcal{C}^\odot$, the *mirror* of $\mathcal{C}$, as follows. The objects of $\mathcal{C}^\odot$ are symbols $C^\odot$, corresponding bijectively to the objects C of $\mathcal{C}$, and the morphisms from $C^\odot$ to $D^\odot$ in $\mathcal{C}^\odot$ are symbols $\alpha^\odot$ in bijective correspondence with the morphisms α from C to D in $\mathcal{C}$. Thus for each morphism

$$\alpha : C \longrightarrow D \text{ in } \mathcal{C},$$

there is exactly one morphism

$$\alpha^\odot : C^\odot \longrightarrow D^\odot \text{ in } \mathcal{C}^\odot.$$

The product in $\mathcal{C}^{\odot}$ is defined by putting

$$\beta^{\odot}\alpha^{\odot} = (\alpha\beta)^{\odot},$$

where $\beta : D \to E$ is a morphism in $\mathcal{C}$ that is composable with α.

Observe that $\mathcal{C}$ and $\mathcal{C}^{\odot}$ are genuinely distinct categories, rather than the same category with two notational conventions, since the order of a pair of morphisms in their product is specified by the axioms.

It is clear that given a right category $\mathcal{D}$, we can construct a left category $\mathcal{D}^{\odot}$ by a similar method, and that there is a formal identification of $\mathcal{C}^{\odot\odot}$ with $\mathcal{C}$ for any category $\mathcal{C}$.

In [BK: IRM] (1.2.6), we introduced a method of turning a left R-module M into a right R°-module M°, where R° is the opposite ring to R. A typical element of R° has the form r° for r in R, and addition and multiplication in the opposite ring are given by

$$r^{\circ} + s^{\circ} = (r+s)^{\circ} \quad \text{and} \quad r^{\circ}s^{\circ} = (sr)^{\circ}.$$

The elements m° of M° are in bijective correspondence with those of M, and the addition and scalar multiplication for M° are defined by

$$m^{\circ} + n^{\circ} = (m+n)^{\circ} \quad \text{and} \quad m^{\circ}r^{\circ} = (rm)^{\circ}.$$

If $\alpha : M \to N$ is a homomorphism of left R-modules, we define a homomorphism α° of right R°-modules by

$$\alpha^{\circ}m^{\circ} = (m\alpha)^{\circ}.$$

Thus we have an identification

$$({}_R\mathcal{M}\text{OD})^{\odot} = \mathcal{M}\text{OD}_{(R^{\circ})},$$

which shows that the use of the mirror category is a generalization of the use of the opposite ring. This identity was used implicitly in Exercise 1.2.14 of [BK: IRM], where we explored the consequences of this technique for switching between left R-modules and right R°-modules. Some authors use this method to avoid dealing both with left and with right modules.

Suppose now that the ring R is commutative. Then $R = R^{\circ}$, so that $({}_R\mathcal{M}\text{OD})^{\odot} = \mathcal{M}\text{OD}_R$. Again, this identity is used implicitly to write scalars and operators on the same side in many elementary textbooks that are concerned only with commutative rings.

We can now elucidate the comment in (1.1.2) about the category $\mathcal{A}\text{B}$ of abelian groups. For, we regard $\mathcal{A}\text{B}$ as the right category $\mathcal{M}\text{OD}_{\mathbb{Z}}$, which is the mirror of the category ${}_{\mathbb{Z}}\mathcal{M}\text{OD}$ of left $\mathbb{Z}$-modules.

We remind the reader that if a category has no obvious chirality, for example, $\mathcal{G}_{\mathcal{P}}$ or $\mathcal{R}_{\mathcal{ING}}$, we always take it to be a right category. However, many group theorists prefer to view $\mathcal{G}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{B}}$ as left categories.

Remark. Some authorities in category theory do not regard a category and its mirror as distinct categories, but simply as alternative notations for composition of morphisms in the one category. From this point of view, a left R-module and the corresponding right R°-module are the same object, in differing notations. However, a ring and its opposite are definitely different rings, as can be seen from the use of opposites in Brauer theory ([Cohn 1979], §10.3). We therefore feel that the distinction between a ring and its opposite should be extended to that between a category and its mirror.

1.1.6 The opposite category

We now give the definition of the *opposite* category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$. This is distinct from the notion of the mirror category that we introduced above.

Given a right category $\mathcal{C}$, the objects C^{op} of the right category $\mathcal{C}^{\mathrm{op}}$ are in bijective correspondence with the objects C of $\mathcal{C}$, and for each pair of objects C^{op}, D^{op} of $\mathcal{C}^{\mathrm{op}}$, there is a bijection between the morphisms α in $\mathrm{Mor}_{\mathcal{C}}(C, D)$ and the morphisms α^{op} in $\mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(D^{\mathrm{op}}, C^{\mathrm{op}})$; thus $\alpha : C \to D$ corresponds to $\alpha^{\mathrm{op}} : D^{\mathrm{op}} \to C^{\mathrm{op}}$. If $\beta : D \to E$ is a morphism in $\mathcal{C}$, then composition in $\mathcal{C}^{\mathrm{op}}$ is given by the rule

$$(\alpha^{\mathrm{op}})(\beta^{\mathrm{op}}) = (\beta\alpha)^{\mathrm{op}}.$$

(The definition for left categories is left to the reader, who, we are confident, will get it right ...)

Note that $\mathcal{C}^{\mathrm{op}}$ has the same chirality as $\mathcal{C}$.

1.1.7 The principle of duality

If we can make a definition or state a result by using only the objects and morphisms of a category $\mathcal{C}$, then we obtain a *dual* definition or result in the opposite category $\mathcal{C}^{\mathrm{op}}$ in which objects $C, D, \ldots$ of $\mathcal{C}$ are replaced by the corresponding objects $C^{\mathrm{op}}, D^{\mathrm{op}}, \ldots$ of $\mathcal{C}^{\mathrm{op}}$ and similarly morphisms $\alpha, \beta, \ldots$ are replaced by $\alpha^{\mathrm{op}}, \beta^{\mathrm{op}}, \ldots$. An example is provided by the relationship between the definitions of projective and injective modules.

A projective module may be defined as a right R-module P for which the following holds.

Pro. Given any surjective R-module homomorphism $\pi : M \to P$, there is a splitting homomorphism $\sigma : P \to M$, that is, $\pi\sigma = id_P$.

On the other hand, a right R-module I is injective if instead we have the following.

Inj. Given any injective R-module homomorphism $\mu : I \to M$, there is a splitting homomorphism $\rho : M \to I$, that is, $\rho\mu = id_I$.

(More details are given in §2.5 of [BK: IRM].)

It is reasonable to assume that if we can generalize surjective and injective homomorphisms of modules to abstract categories, then a surjective map in $\mathcal{C}$ will correspond to an injective map in the opposite category $\mathcal{C}^{\text{op}}$, and so a projective object in $\mathcal{C}$ will correspond to an injective object in the opposite category. (This assumption is justified in the next chapter – see (2.2.2).) Thus, provided that we confine ourselves to statements that involve only objects and morphisms, we need only prove results about projective objects, since the corresponding results for injective objects are then true by duality.

This technique for pairing together definitions, arguments, etc. in a category and its opposite is the *principle of duality*, and the phrase *by duality* is used to indicate this method of argument.

1.1.8 Subcategories

A *subcategory* $\mathcal{D}$ of a (right) category $\mathcal{C}$ is defined by the following data.

Sub 1. A subclass $\text{Ob}\,\mathcal{D}$ of $\text{Ob}\,\mathcal{C}$, which specifies the objects that belong to $\mathcal{D}$.

Sub 2. For each pair of objects C, D of $\mathcal{D}$, a subset $\text{Mor}_{\mathcal{D}}(C,D)$ of $\text{Mor}_{\mathcal{C}}(C,D)$; these are the morphisms from C to D in $\mathcal{D}$.

Sub 3. If C is an object of $\mathcal{D}$, then the identity morphism id_C in $\text{Mor}_{\mathcal{C}}(C,C)$ shall belong to $\text{Mor}_{\mathcal{D}}(C,C)$ (where it is again the identity morphism).

Sub 4. If $\alpha \in \text{Mor}_{\mathcal{D}}(C,D)$ and $\beta \in \text{Mor}_{\mathcal{D}}(D,E)$, then

$$\beta\alpha \in \text{Mor}_{\mathcal{D}}(C,E),$$

(where the product is the product in $\mathcal{C}$).

It is clear that a subcategory of a right category is itself a right category with identities and products inherited from $\mathcal{C}$.

The corresponding definitions for a left category are obvious.

Examples of subcategories are $\mathcal{A}\textsc{b}$ in $\mathcal{G}\textsc{p}$ and $\mathcal{R}\textsc{ing}$ in $\mathcal{R}\textsc{ng}$. Note that any group homomorphism from G to H is also a morphism in $\mathcal{A}\textsc{b}$ if the groups G and H happen to be abelian. In contrast, a morphism in $\mathcal{R}\textsc{ng}$ from a ring R to a ring S need not be allowable as a morphism in $\mathcal{R}\textsc{ing}$. This can be seen by computing $\mathrm{Mor}(0, S)$ in the two cases: if S is a ring other than the zero ring, then $\mathrm{Mor}(0, S) = \{0\}$ in the first case but $\mathrm{Mor}(0, S) = \emptyset$ in the second.

1.1.9 Full subcategories

A *full* subcategory $\mathcal{D}$ of $\mathcal{C}$ is one in which

$$\mathrm{Mor}_{\mathcal{D}}(C, D) = \mathrm{Mor}_{\mathcal{C}}(C, D)$$

for any two objects of $\mathcal{D}$.

It is obvious that any subclass $\mathcal{D}$ of the class of objects of $\mathcal{C}$ gives rise to a unique full subcategory of $\mathcal{C}$.

This method of constructing subcategories of $\mathcal{M}\text{OD}_R$, for various rings R, will be very useful to us in the sequel. For this reason, we define a *module category* to be any category which is a full subcategory of $\mathcal{M}\text{OD}_R$ (or ${}_R\mathcal{M}\text{OD}$) for some ring R.

Here are some important examples.

$\mathcal{F}\text{REE}_R$, the category of all free right R-modules.
$\mathcal{P}\text{ROJ}_R$, the category of all projective right R-modules.
$\mathcal{M}_R$, the category of finitely generated right R-modules.
$\mathcal{F}_R$, the category of free right modules R^k of finite rank.
$\mathcal{P}_R$, the category of finitely generated projective right R-modules.

The corresponding subcategories of ${}_R\mathcal{M}\text{OD}$ are denoted ${}_R\mathcal{F}\text{REE}$, ${}_R\mathcal{P}\text{ROJ}$, ${}_R\mathcal{M}$, ${}_R\mathcal{F}$ and ${}_R\mathcal{P}$ respectively.

1.1.10 Some remarks on set theory and small categories

In our definition of a category, we use the expression '*class*', rather than the word '*set*' which the reader might have expected. The reason for this is that we wish to make a naïve distinction between sets, on which all mathematical constructions are allowed, and 'non-sets', which are too large to permit some operations. The aim is to avoid Russell's Paradox and similar traps: if X is a set of sets, it is sensible to ask whether or not the set X is itself in X. The paradox arises by taking X to be the set of all sets that do not contain themselves. Our avoiding action is to declare that this particular choice of X is a class but not a set; essentially, X is too large to be considered to be a set. The kind of set theory that we have in mind is expanded in detail in [Herrlich & Strecker 1979], [van Dalen, Doets & de Swart 1979], and [Levy 1979].

If the class $\mathrm{Ob}\,\mathcal{C}$ of objects of the category $\mathcal{C}$ is in fact a set, then the category is said to be *small*.