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978-0-521-63274-4 - An Introduction to Rings and Modules with K-theory in View

A. J. Berrick and M. E. Keating

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