

Cambridge University Press

978-0-521-63204-1 - Infinite-Dimensional Dynamical Systems: An Introduction to
Dissipative Parabolic PDEs and the Theory of Global Attractors

James C. Robinson

Table of Contents

[More information](#)

Contents

| | |
|--|----------------|
| Preface | <i>page</i> xv |
| Introduction | 1 |
| Part I Functional Analysis | 9 |
| 1 Banach and Hilbert Spaces | 11 |
| 1.1 Banach Spaces and Some General Topology | 11 |
| 1.2 The Euclidean Space \mathbb{R}^m | 12 |
| 1.3 The Spaces C^r and $C^{r,\gamma}$ of Continuous Functions | 14 |
| 1.3.1 Mollification and Approximation by Smooth Functions | 18 |
| 1.4 The L^p Spaces of Lebesgue Integrable Functions | 20 |
| 1.4.1 Lebesgue Integration | 20 |
| 1.4.2 The Lebesgue Spaces $L^p(\Omega)$ with $1 \leq p < \infty$ | 22 |
| 1.4.3 The Lebesgue Space $L^\infty(\Omega)$ | 28 |
| 1.4.4 The Spaces $L^p_{loc}(\Omega)$ of Locally Integrable Functions | 30 |
| 1.4.5 The l^p Sequence Spaces, $1 \leq p \leq \infty$ | 32 |
| 1.5 Hilbert Spaces | 33 |
| 1.5.1 The Orthogonal Projection onto a Linear Subspace | 35 |
| 1.5.2 Bases in Hilbert Spaces | 36 |
| 1.5.3 Noncompactness of the Unit Ball | 38 |
| Exercises | 39 |
| Notes | 41 |
| 2 Ordinary Differential Equations | 42 |
| 2.1 Existence and Uniqueness – A Fixed-Point Method | 43 |
| 2.1.1 The Contraction Mapping Theorem | 44 |
| 2.1.2 Local Existence for Lipschitz f | 45 |

Cambridge University Press

978-0-521-63204-1 - Infinite-Dimensional Dynamical Systems: An Introduction to
Dissipative Parabolic PDEs and the Theory of Global Attractors

James C. Robinson

Table of Contents

[More information](#)

| | | |
|----------|---|-----------|
| viii | <i>Contents</i> | |
| 2.2 | Global Existence | 48 |
| 2.3 | Existence but No Uniqueness – An Approximation Method | 49 |
| 2.3.1 | The Arzelà–Ascoli Theorem | 49 |
| 2.3.2 | Local Existence for Continuous f | 51 |
| 2.4 | Differential Inequalities | 53 |
| 2.5 | Continuous Dependence on Initial Conditions | 56 |
| 2.6 | Conclusion | 59 |
| | Exercises | 59 |
| | Notes | 61 |
| 3 | Linear Operators | 62 |
| 3.1 | Bounded Linear Operators on Banach Spaces | 62 |
| 3.2 | Domain, Range, Kernel, and the Inverse Operator | 65 |
| 3.3 | The Baire Category Theorem | 66 |
| 3.4 | Compact Operators | 68 |
| 3.5 | Compact Symmetric Operators on Hilbert Spaces | 72 |
| 3.6 | Obtaining an Eigenbasis from a Compact Symmetric Operator | 74 |
| 3.7 | Unbounded Operators | 79 |
| 3.8 | Extensions and Closable Operators | 80 |
| 3.9 | Spectral Theory for Unbounded Symmetric Operators | 81 |
| 3.10 | Positive Operators and Their Fractional Powers | 83 |
| | Exercises | 85 |
| | Notes | 87 |
| 4 | Dual Spaces | 89 |
| 4.1 | The Hahn–Banach Theorem | 89 |
| 4.2 | Examples of Dual Spaces | 93 |
| 4.2.1 | The Dual Space of L^p , $1 < p < \infty$ | 93 |
| 4.2.2 | The Dual Space of l^p , $1 < p < \infty$ | 94 |
| 4.2.3 | The Dual Spaces of L^1 and L^∞ | 96 |
| 4.2.4 | The Dual Space of l^1 and l^∞ | 96 |
| 4.3 | Dual Spaces of Hilbert Spaces | 99 |
| 4.4 | Reflexive Spaces | 100 |
| 4.5 | Notions of Weak Convergence | 101 |
| 4.5.1 | Weak Convergence | 101 |
| 4.5.2 | Weak-* Convergence | 104 |
| 4.6 | The Alaoglu Weak-* Compactness Theorem | 105 |
| | Exercises | 107 |
| | Notes | 107 |

Cambridge University Press

978-0-521-63204-1 - Infinite-Dimensional Dynamical Systems: An Introduction to
Dissipative Parabolic PDEs and the Theory of Global Attractors

James C. Robinson

Table of Contents

[More information](#)

| <i>Contents</i> | | ix |
|--|--|------------|
| 5 | Sobolev Spaces | 109 |
| 5.1 | Generalised Notions of Derivatives | 109 |
| 5.1.1 | The Weak Derivative | 109 |
| 5.1.2 | The Distribution Derivative | 111 |
| 5.2 | General Sobolev Spaces | 114 |
| 5.2.1 | Sobolev Spaces and the Closure of Differential Operators | 115 |
| 5.2.2 | The Hilbert Space $H^k(\Omega)$ | 115 |
| 5.3 | Outline of the Rest of the Chapter | 119 |
| 5.4 | $C^\infty(\Omega)$ is Dense in $H^k(\Omega)$ | 120 |
| 5.5 | An Extension Theorem | 124 |
| 5.5.1 | Extending Functions in $H^k(\mathbb{R}_m^+)$ | 125 |
| 5.5.2 | Coordinate Changes | 127 |
| 5.5.3 | Straightening the Boundary | 128 |
| 5.5.4 | Extending Functions in $H^k(\Omega)$ | 129 |
| 5.6 | Density of $C^\infty(\overline{\Omega})$ in $H^k(\Omega)$ | 131 |
| 5.7 | The Sobolev Embedding Theorem – H^k , C^r , and L^p | 132 |
| 5.7.1 | Integrability of Functions in Sobolev Spaces | 132 |
| 5.7.2 | Sobolev Spaces and Spaces of Continuous Functions | 139 |
| 5.7.3 | The Sobolev Embedding Theorem | 142 |
| 5.8 | A Compactness Theorem | 143 |
| 5.9 | Boundary Values | 145 |
| 5.10 | Sobolev Spaces of Periodic Functions | 149 |
| | Exercises | 152 |
| | Notes | 153 |
| Part II Existence and Uniqueness Theory | | 157 |
| 6 | The Laplacian | 159 |
| 6.1 | Classical, Strong, and Weak Solutions | 160 |
| 6.2 | Weak Solutions of Poisson's Equation | 160 |
| 6.3 | Higher Regularity for the Laplacian I: Periodic Boundary Conditions | 164 |
| 6.4 | Higher Regularity for the Laplacian II: Dirichlet Boundary Conditions | 168 |
| 6.4.1 | A Heuristic Estimate | 168 |
| 6.4.2 | Difference Quotients | 170 |
| 6.4.3 | Interior Regularity Result | 172 |
| 6.5 | Boundary Regularity for the Laplacian | 175 |
| 6.5.1 | Regularity up to a Flat Boundary | 175 |
| 6.5.2 | Regularity up to a C^2 Boundary | 180 |

Cambridge University Press

978-0-521-63204-1 - Infinite-Dimensional Dynamical Systems: An Introduction to
Dissipative Parabolic PDEs and the Theory of Global Attractors

James C. Robinson

Table of Contents

[More information](#)

| x | <i>Contents</i> | |
|----------|--|------------|
| 6.5.3 | $H^{2k}(\Omega)$ and Domains of A^k | 183 |
| | Exercises | 185 |
| | Notes | 187 |
| 7 | Weak Solutions of Linear Parabolic Equations | 188 |
| 7.1 | Banach-Space Valued Function Spaces | 188 |
| 7.2 | Weak Solutions of Parabolic Equations | 194 |
| 7.3 | The Galerkin Method: Truncated Eigenfunction Expansions | 197 |
| 7.4 | Weak Solutions | 200 |
| 7.4.1 | The Galerkin Approximations | 201 |
| 7.4.2 | Uniform Bounds on u_n in Various Spaces | 202 |
| 7.4.3 | Extraction of an Appropriate Subsequence | 203 |
| 7.4.4 | Properties of the Weak Solution | 205 |
| 7.4.5 | Uniqueness and Continuous Dependence on Initial Conditions | 206 |
| 7.5 | Strong Solutions | 206 |
| 7.6 | Higher Regularity: Spatial and Temporal | 209 |
| | Exercises | 210 |
| | Notes | 212 |
| 8 | Nonlinear Reaction–Diffusion Equations | 213 |
| 8.1 | Results to Deal with the Nonlinear Term | 214 |
| 8.1.1 | A Compactness Theorem | 214 |
| 8.1.2 | A Weak Version of the Dominated Convergence Theorem | 217 |
| 8.2 | The Basis for the Galerkin Expansion | 219 |
| 8.3 | Weak Solutions | 221 |
| 8.3.1 | A Semidynamical System on $L^2(\Omega)$ | 226 |
| 8.4 | Strong Solutions | 227 |
| | Exercises | 231 |
| | Notes | 232 |
| 9 | The Navier–Stokes Equations: Existence and Uniqueness | 234 |
| 9.1 | The Stokes Operator | 235 |
| 9.2 | The Weak Form of the Navier–Stokes Equation | 239 |
| 9.3 | Properties of the Trilinear Form | 241 |
| 9.4 | Existence of Weak Solutions | 244 |
| 9.5 | Unique Weak Solutions in Two Dimensions | 250 |
| 9.6 | Existence of Strong Solutions in Two Dimensions | 252 |
| 9.7 | Uniqueness of 3D Strong Solutions | 255 |
| 9.8 | Dynamical Systems Generated by the 2D Equations | 256 |

Cambridge University Press

978-0-521-63204-1 - Infinite-Dimensional Dynamical Systems: An Introduction to
Dissipative Parabolic PDEs and the Theory of Global Attractors

James C. Robinson

Table of Contents

[More information](#)

| <i>Contents</i> | | xi |
|--|---|------------|
| | Exercises | 256 |
| | Notes | 257 |
| Part III Finite-Dimensional Global Attractors | | 259 |
| 10 | The Global Attractor: Existence and General Properties | 261 |
| 10.1 | Semigroups | 261 |
| 10.2 | Dissipation | 262 |
| 10.3 | Limit Sets and Attractors | 265 |
| 10.3.1 | Limit Sets | 265 |
| 10.3.2 | The Global Attractor | 266 |
| 10.4 | A Theorem for the Existence of Global Attractors | 269 |
| 10.5 | An Example – The Lorenz Equations | 271 |
| 10.6 | Structure of the Attractor | 272 |
| 10.6.1 | Gradient Systems and Lyapunov Functions | 274 |
| 10.7 | How the Attractor Determines the Asymptotic Dynamics | 276 |
| 10.8 | Continuity Properties of the Attractor | 278 |
| 10.8.1 | Upper Semicontinuity | 278 |
| 10.8.2 | Lower Semicontinuity | 279 |
| 10.9 | Conclusion | 280 |
| | Exercises | 281 |
| | Notes | 282 |
| 11 | The Global Attractor for Reaction–Diffusion Equations | 285 |
| 11.1 | Absorbing Sets and the Attractor | 285 |
| 11.1.1 | An Absorbing Set in L^2 | 286 |
| 11.1.2 | An Absorbing Set in H_0^1 | 287 |
| 11.1.3 | The Global Attractor | 290 |
| 11.2 | Regularity Results | 290 |
| 11.2.1 | A Bound in L^∞ | 290 |
| 11.2.2 | A Bound in $H^2(\Omega)$ | 293 |
| 11.2.3 | Further Regularity | 295 |
| 11.3 | Injectivity on \mathcal{A} | 296 |
| 11.4 | A Lyapunov Functional | 299 |
| 11.5 | The Chaffee–Infante Equation | 301 |
| 11.5.1 | Stationary Points | 301 |
| 11.5.2 | Bifurcations around the Zero State | 304 |
| | Exercises | 306 |
| | Notes | 307 |

Cambridge University Press

978-0-521-63204-1 - Infinite-Dimensional Dynamical Systems: An Introduction to
Dissipative Parabolic PDEs and the Theory of Global Attractors

James C. Robinson

Table of Contents

[More information](#)

| | | |
|----------------|---|------------|
| xii | <i>Contents</i> | |
| 12 | The Global Attractor for the Navier–Stokes Equations | 309 |
| 12.1 | 2D Navier–Stokes Equations | 309 |
| 12.1.1 | An Absorbing Set in \mathbb{L}^2 | 310 |
| 12.1.2 | An Absorbing Set in \mathbb{H}^1 | 311 |
| 12.1.3 | An Absorbing Set in \mathbb{H}^2 | 313 |
| 12.1.4 | Comparison of the Attractors in H and V and Further Regularity Results | 315 |
| 12.1.5 | Injectivity on the Attractor | 316 |
| 12.2 | The 3D Navier–Stokes Equations | 317 |
| 12.2.1 | An Absorbing Set in V | 318 |
| 12.2.2 | An Absorbing Set in $D(A)$ and a Global Attractor | 322 |
| 12.3 | Conclusion | 323 |
| | Exercises | 323 |
| | Notes | 324 |
| 13 | Finite-Dimensional Attractors: Theory and Examples | 325 |
| 13.1 | Measures of Dimension | 326 |
| 13.1.1 | The “Fractal” Dimension | 326 |
| 13.1.2 | The Hausdorff Dimension | 330 |
| 13.1.3 | Hausdorff versus Fractal Dimension | 334 |
| 13.2 | Bounding the Attractor Dimension Dynamically | 336 |
| 13.3 | Example I: The Reaction–Diffusion Equation | 343 |
| 13.3.1 | Uniform Differentiability | 343 |
| 13.3.2 | A Bound on the Attractor Dimension | 346 |
| 13.4 | Example II: The 2D Navier–Stokes Equations | 347 |
| 13.4.1 | Uniform Differentiability | 347 |
| 13.4.2 | A Bound on the Attractor Dimension | 349 |
| 13.5 | Physical Interpretation of the Attractor Dimension | 350 |
| 13.6 | Conclusion | 352 |
| | Exercises | 352 |
| | Notes | 354 |
| Part IV | Finite-Dimensional Dynamics | 357 |
| 14 | The Squeezing Property: Determining Modes | 359 |
| 14.1 | The Squeezing Property | 359 |
| 14.2 | An Approximate Manifold Structure for \mathcal{A} | 360 |
| 14.3 | Determining Modes | 363 |
| 14.4 | The Squeezing Property for Reaction–Diffusion Equations | 365 |
| 14.5 | The 2D Navier–Stokes Equations | 369 |

Cambridge University Press

978-0-521-63204-1 - Infinite-Dimensional Dynamical Systems: An Introduction to
Dissipative Parabolic PDEs and the Theory of Global Attractors

James C. Robinson

Table of Contents

[More information](#)

| <i>Contents</i> | | xiii |
|-----------------|--|------------|
| 14.5.1 | Checking the Squeezing Property | 369 |
| 14.5.2 | Approximate Inertial Manifolds | 371 |
| 14.6 | Finite-Dimensional Exponential Attractors | 374 |
| 14.7 | Conclusion | 379 |
| | Exercises | 379 |
| | Notes | 383 |
| 15 | The Strong Squeezing Property: Inertial Manifolds | 385 |
| 15.1 | Inertial Manifolds and “Slaving” | 385 |
| 15.2 | A Geometric Existence Proof | 388 |
| 15.2.1 | The Strong Squeezing Property | 388 |
| 15.2.2 | The Existence Proof | 391 |
| 15.3 | Finding Conditions for the Strong Squeezing Property | 394 |
| 15.4 | Inertial Manifolds for Reaction–Diffusion Equations | 396 |
| 15.4.1 | Preparing the Equation | 396 |
| 15.4.2 | Checking the Spectral Gap Condition | 399 |
| 15.4.3 | Extensions to Other Domains and Higher Dimensions | 400 |
| 15.5 | More General Conditions for the Strong Squeezing Property | 401 |
| 15.5.1 | Inertial Manifolds and the Navier–Stokes Equations | 401 |
| 15.6 | Conclusion | 403 |
| | Exercises | 403 |
| | Notes | 404 |
| 16 | A Direct Approach | 406 |
| 16.1 | Parametrising the Attractor | 407 |
| 16.1.1 | Experimental Measurements as Parameters | 411 |
| 16.2 | An Extension Theorem | 411 |
| 16.3 | Embedding the Dynamics Without Uniqueness | 412 |
| 16.3.1 | Continuity of F on \mathcal{A} for the Scalar Reaction–Diffusion Equation | 414 |
| 16.3.2 | Continuity of F on \mathcal{A} for the 2D Navier–Stokes Equations | 415 |
| 16.4 | A Discrete-Time Utopian Theorem | 415 |
| 16.4.1 | The Topology of Global Attractors | 416 |
| 16.4.2 | The “within ϵ ” Discrete Utopian Theorem | 419 |
| 16.5 | Conclusion | 422 |
| | Exercises | 422 |
| | Notes | 424 |
| 17 | The Kuramoto–Sivashinsky Equation | 426 |
| 17.1 | Preliminaries | 426 |

Cambridge University Press

978-0-521-63204-1 - Infinite-Dimensional Dynamical Systems: An Introduction to
Dissipative Parabolic PDEs and the Theory of Global Attractors

James C. Robinson

Table of Contents

[More information](#)

| | | |
|-------|---|------------|
| xiv | <i>Contents</i> | |
| 17.2 | Existence and Uniqueness of Solutions | 428 |
| 17.3 | Absorbing Sets and the Global Attractor | 429 |
| 17.4 | The Attractor is Finite-Dimensional | 431 |
| 17.5 | Inertial Manifolds | 432 |
| | Notes | 433 |
| | Appendix A Sobolev Spaces of Periodic Functions | 435 |
| A.1 | The Sobolev Embedding Theorem – H^s , C^r , and L^p | 435 |
| A.1.1 | Conditions for $H^s(Q) \subset C^0(\overline{Q})$ | 435 |
| A.1.2 | Integrability Properties of Functions in H^s | 436 |
| A.2 | Rellich–Kondrachov Compactness Theorem | 437 |
| | Appendix B Bounding the Fractal Dimension Using the Decay of Volume Elements | 439 |
| | References | 445 |
| | Index | 453 |