Infinite-Dimensional Dynamical Systems

This book develops the theory of global attractors for a class of parabolic PDEs that includes reaction–diffusion equations and the Navier–Stokes equations, two examples that are treated in detail. A lengthy chapter on Sobolev spaces provides the framework that allows a rigorous treatment of existence and uniqueness of solutions for both linear time-independent problems (Poisson's equation) and the nonlinear evolution equations that generate the infinite-dimensional dynamical systems of the title. Attention then turns to the global attractor, a finite-dimensional subset of the infinite-dimensional phase space that determines the asymptotic dynamics. In particular, the concluding chapters investigate in what sense the dynamics restricted to the attractor are themselves "finite-dimensional."

The book is intended as a didactic text for first-year graduate students and assumes only a basic knowledge of elementary functional analysis.

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Infinite-Dimensional Dynamical Systems

An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors

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Preface

Inspired by the success of dynamical systems theory in treating finite-dimensional systems, a similar approach has been developed for partial differential equations over the past two decades. In particular, this book focuses on the way in which the long-term dynamics of certain models can be studied by means of their global attractor, and it aims to give a didactic introduction to the subject at a level suitable for a first-year UK graduate student.

As such, there are various disparate topics that need to be covered, and I have attempted to provide what I hope is more than just an "introduction" to the theory of global attractors by giving a systematic development of the subject from its most basic foundations. There are, however, many excellent texts that treat some of these topics individually, and I would particularly like to mention Renardy & Rogers (1992), Evans (1998), and Gilbarg & Trudinger (1983), which have all had a large influence on the organisation of material in the earlier chapters, and Temam (1988), which contains, in some form, much of the material in later chapters.* Finally, Doering & Gibbon (1995) cover some of the same material in the context of the Navier–Stokes equations, and this book can be viewed in part as a retelling of their results in a language applicable to many other examples.

When I started writing I was planning to produce the kind of book that I would like to have read when I started my Ph.D.; I hope that, in the end, I have managed to produce the sort of text that I should have read. Although my natural inclination was to assume the minimum analytical background (having in mind my own inadequacies as a graduate student), considerations of space

* His book, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, has influenced my choice of title, even though both books treat only a subset of such systems. The theory of global attractors relies on some kind of dissipation, automatically excluding Hamiltonian systems, for example. The main focus here is on the Navier–Stokes equations and related models.

have forced me to assume some previous familiarity with the theory of Lebesgue integration, of which just a brief outline is given in Chapter 1. Since knowledge of general measure theory is not necessary, a readable treatment of integration such as that given in Priestley (1997) is more than adequate.

The first part of the book covers the functional analysis needed throughout all that follows. Chapters 1–4, which concern Banach and Hilbert spaces, existence and uniqueness theory for ordinary differential equations, linear maps, spectral theory for compact symmetric operators, dual spaces, and weak convergence, may be revision for many readers. This section concludes with a lengthy chapter that treats Sobolev spaces, a cornerstone of the theory, in detail.

Chapter 6 treats questions of existence, uniqueness, and regularity for Poisson's equation. Although restricted to this case for ease of presentation, the analysis there can easily be adapted to more general elliptic equations.

From here on all the machinery is in place to consider existence and uniqueness for time-dependent equations. Chapter 7 introduces the Galerkin method as a means of proving these properties, using a linear parabolic equation as an example. These ideas are then applied in Chapter 8 to a scalar reaction–diffusion equation and in Chapter 9 to the two-dimensional Navier–Stokes equations with periodic boundary conditions.

Chapters 10–12 introduce the global attractor and show how to prove its existence for the two examples of Chapters 8 and 9. Chapter 13 defines the fractal and Hausdorff measures of dimension and provides a method to estimate these dimensions for global attractors, which is then applied to our two examples.

The next three chapters investigate how the finite dimensionality of the attractor affects the asymptotic dynamics. Chapter 14 concerns the squeezing property, covering "determining modes," approximate inertial manifolds, and exponential attractors. Chapter 15 treats the theory of inertial manifolds, using the geometric "strong squeezing property" as a basis for the analysis. Chapter 16 shows that the attractor can be parametrised using a finite number of parameters, and it gives a proof that there is a finite-dimensional system that reproduces the attractor dynamics.

The final chapter consists of a series of exercises that apply many of the techniques learned throughout the book to an analysis of the Kuramoto–Sivashinsky equation.

It is a peculiarity of the British tradition that the rigorous study of partial differential equations is not a common element in the syllabus of many mathematics departments. Even to students with a strong background in functional analysis the material in Chapters 5–9 is likely to be new. As such, teaching from this book requires some judicious editing of material. When I gave one of the lecture courses from which this book is partly derived, I restricted attention to Sobolev spaces of periodic functions (the majority of Chapter 5 can then be replaced with Appendix A) and similarly avoided the many calculations involved in the regularity theory of the Laplacian in general domains. In this way one can get on to the more interesting questions raised in the second half of this book at an earlier stage than would be possible if treating Sobolev spaces and elliptic regularity in detail.

Many people have been helpful, knowingly and unknowingly, in the creation of this book over the past four years. Thomas Ransford's elegant Cambridge Part III lecture course on distribution theory was revelatory and completely changed my view of pure mathematics, while Colin Sparrow's course on dynamical systems was equally inspirational. Paul Glendinning, my Ph.D. supervisor, was a great encouragement as I then tried to combine my two new enthusiasms.

Some years later I was very fortunate to have the opportunity of lecturing on some of the material here as a Part III course in two consecutive years, and without David Crighton's initial encouragement to turn my lecture notes into a book I would not even have started this long task. Rebecca Hoyle managed to sit through the first of these courses (of which thankfully little trace remains here) and was extremely supportive throughout. A kind invitation from José Langa, Tomás Caraballo, and Enrique Fernández Cara to give a series of lectures at the University of Seville over the Easter period in 1997 was the germ of Chapters 9–16; during my visit I enjoyed unparalleled hospitality, particularly from José.

Ciprian Foias, John Gibbon, Edriss Titi, and Roger Temam have all been continually inspiring and maintained my enthusiasm for the subject as a whole, and Brian Hunt very kindly provided me with a proof I needed for Chapter 13, which is reproduced in Appendix B.

Tania, who has been very patient as my head has filled up with Sobolev spaces, has made me laugh and smile for the past three years; while Daisy, our cat, has sat repeatedly on whichever pile of papers I most needed, this particular talent keeping me cheerful throughout the final weeks of constant revision.

Finally, my utmost gratitude goes to Peter Friz and Robert Mackay, both of whom read through the manuscript very closely and critically, and without whom there would be many more errors and inconsistencies than those which undoubtedly remain. The responsibility for these is, sadly, entirely my own.