Holomorphic Spaces
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Holomorphic Spaces: A Brief and Selective Survey

DONALD SARASON

Abstract. This article traces several prominent trends in the development of the subject of holomorphic spaces, with emphasis on operator-theoretic aspects.

The term "Holomorphic Spaces," the title of a program held at the Mathematical Sciences Research Institute in the fall semester of 1995, is short for "Spaces of Holomorphic Functions." It refers not so much to a branch of mathematics as to a common thread running through much of modern analysis—through functional analysis, operator theory, harmonic analysis, and, of course, complex analysis. This article will briefly outline the development of the subject from its origins in the early 1900's to the present, with a bias toward operator-theoretic aspects, in keeping with the main emphasis of the MSRI program. I hope that the article will be accessible not only to workers in the field but to analysts in general.

Origins

The subject began with the thesis of P. Fatou [1906], a student of H. Lebesgue. The thesis is a study of the boundary behavior of certain harmonic functions in the unit disk (those representable as Poisson integrals). It contains a proof, for example, that a bounded holomorphic function in the disk has a nontangential limit at almost every point of the unit circle. This initial link between function theory on the circle (real analysis) and function theory in the disk (complex analysis) recurred continually in the ensuing years. Some of the highlights are the paper of F. Riesz and M. Riesz [1916] on the absolute continuity of analytic measures; F. Riesz's paper [1923] in which he christened the Hardy spaces, H^p, and introduced the technique of dividing out zeros (i.e., factoring by a Blaschke product); G. Szegö's investigations [1920; 1921] of Toeplitz forms; M. Riesz's proof [1924] of the L^p boundedness of the conjugation operator ($1 < p < \infty$);

A. N. Kolmogorov's proof [1925] of the weak-L^1 boundedness of the conjugation operator; G. H. Hardy and J. E. Littlewood's introduction of their maximal function [1930]; and R. Nevanlinna's development [1936] of his theory of functions of bounded characteristic. By the late 1930's the theory had expanded to the point where it could become the subject of a monograph. The well-known book of I. I. Privalov appeared in 1941, the year of the author's death, and was republished nine years later [Privalov 1950], followed by a German translation [Privalov 1956].

Beurling's Paper

Simultaneously, the general theory of Banach spaces and their operators had been developing. By mid-century, when A. Beurling published a seminal paper [1949], the time was ripe for mutual infusion. After posing two general questions about Hilbert space operators with complete sets of eigenvectors, Beurling's paper focuses on the (closed) invariant subspaces of the unilateral shift operator on the Hardy space H^2 of the unit disk (an operator whose adjoint is of the kind just mentioned).

For the benefit of readers who do not work in the field, here are a few of the basic definitions. For $p > 0$ the Hardy space H^p consists of the holomorphic functions f in the unit disk, $\mathbb{D}$, satisfying the growth condition $\sup_{0<r<1} \|f_r\|_p < \infty$, where f_r is the function on the unit circle defined by $f_r\left(e^{i\theta}\right) = f\left(re^{i\theta}\right)$, and $\|f_r\|_p$ denotes the norm of f_r in the L^p space of normalized Lebesgue measure on the circle, hereafter denoted simply by L^p. As noted earlier, the spaces H^p were introduced by F. Riesz [1923]; they were named by him in honor of G. H. Hardy, who had proved [1915] that $\|f_r\|_p$ increases with r (unless f is constant). From the work of Fatou and his successors one knows that each function in H^p has an associated boundary function, defined almost everywhere on $\partial\mathbb{D}$ in terms of nontangential limits. Because of this, one can identify H^p with a subspace of L^p; in case $p \geq 1$, the subspace in question consists of the functions in L^p whose Fourier coefficients with negative indices vanish (i.e., the functions whose Fourier series are of power series type). A function in H^p, for $p \geq 1$, can be reconstructed from its boundary function by means of the Poisson integral, or the Cauchy integral. The space H^2 can be alternatively described as the space of holomorphic functions in $\mathbb{D}$ whose Taylor coefficients at the origin are square summable. In the obvious way it acquires a Hilbert space structure in which the functions z^n, for $n = 0, 1, 2, \ldots$, form an orthonormal basis.

The unilateral shift is the operator S on H^2 of multiplication by z, the identity function. It is an isometry, sending the n-th basis vector, z^n, to the $(n+1)$-st, z^{n+1}. It is, in fact, the simplest pure isometry. (A Hilbert space isometry is called pure if it has no unitary direct summand. Every pure isometry is a direct sum of copies of S.) Beurling showed that the invariant subspace structure of S mirrors the factorization theory of H^2 functions.

From the work of F. Riesz and Nevanlinna it is known that every nonzero function in H^p can be written as the product of what Beurling called an outer function and an inner function. The factors are unique to within unimodular multiplicative constants. An outer function is a nowhere vanishing holomorphic function f in $\mathbb{D}$ such that $\log |f|$ is the Poisson integral of its boundary function. Beurling showed that the outer functions in H^2 are the cyclic vectors of the operator S (i.e., the functions contained in no proper S-invariant subspaces). An inner function is a function in H^∞ whose boundary function has unit modulus almost everywhere. Beurling showed that if the H^2 function f has the factorization $f = uf_0$, with u an inner function and f_0 an outer function, then the S-invariant subspace generated by f is the same as that generated by u, and it equals uH^2. Finally, Beurling showed that every invariant subspace of S is generated by a single function, and hence by an inner function. Thus, understanding the invariant subspace structure of S is tantamount to understanding the structure of inner functions.

There are two basic kinds of inner functions: Blaschke products and singular functions. Only the constant inner functions are of both kinds, and every inner function is the product of a Blaschke product and a singular function, the factors being unique to within unimodular multiplicative constants. Blaschke products (products of Blaschke factors) are associated with zero sequences. The zero sequence of a function in H^2 (in fact, of a function in any H^p) is a so-called Blaschke sequence, a finite sequence in $\mathbb{D}$ or an infinite sequence $(z_n)_{n=1}^\infty$ satisfying $\sum(1 - |z_n|) < \infty$ (the Blaschke condition). The Blaschke factor corresponding to a point w of $\mathbb{D}$ is, in case $w \neq 0$, the linear-fractional map of $\mathbb{D}$ onto $\mathbb{D}$ that sends w to 0 and 0 to the positive real axis, and in case $w = 0$ it is the identity function. A Blaschke product is the product of the Blaschke factors corresponding to the terms of a Blaschke sequence, or a unimodular constant times such a function. In the case of a finite sequence it is obviously an inner function, and in the case of an infinite sequence, the Blaschke condition is exactly what one needs to prove that the corresponding infinite product of Blaschke factors converges locally uniformly in $\mathbb{D}$ to an inner function. If the inner function associated with an S-invariant subspace is a Blaschke product, then the subspace is just the subspace of functions in H^2 that vanish at the points of the corresponding Blaschke sequence (with the appropriate multiplicities at repeated points).

A singular function is an inner function without zeroes in $\mathbb{D}$. The logarithm of the modulus of such a function, if the function is nonconstant, is a negative harmonic function in $\mathbb{D}$ having the nontangential limit 0 at almost every point of $\partial\mathbb{D}$. One can conclude on the basis of the theory of Poisson integrals that the logarithm of the modulus of a nonconstant singular function is the Poisson integral of a negative singular measure on $\partial\mathbb{D}$. The most general nonconstant singular function can thus be represented as

$$\lambda \exp\left(-\int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu\,(e^{i\theta})\right) \qquad \text{for } z \in \mathbb{D},$$

where λ is a unimodular constant and μ is a positive singular measure on $\partial\mathbb{D}$. The simplest such function is

$$\exp\left(\frac{z+1}{z-1}\right),$$

corresponding to the case where $\lambda = 1$ and μ consists of a unit point mass at the point 1. If the inner function associated with an S-invariant subspace is a singular function, then the functions in the subspace have no common zeroes in $\mathbb{D}$, but the common singular inner factor they share forces them all to have the nontangential limit 0 almost everywhere on $\partial\mathbb{D}$ with respect to the associated singular measure.

Because of Beurling's theorem, the preceding description of inner functions translates into a description of the invariant subspaces of the operator S. The theorem is a splendid early example of how a natural question in operator theory can lead deeply into analysis.

Multiple Shifts and Operator Models

Beurling's work was extended to multiple shifts by P. D. Lax [1959] and P. R. Halmos [1961]. Here one naturally encounters vector-valued function theory.

For $1 \le n \le \aleph_0$, the unilateral shift of multiplicity n (that is, the direct sum of n copies of S) can be modeled as the operator of multiplication by z on a vector-valued version of H^2; the functions in this space have values belonging to an auxiliary Hilbert space $\mathcal{E}$ of dimension n. The space, usually denoted by $H^2(\mathcal{E})$, can be defined, analogously to the scalar case, as the space of holomorphic $\mathcal{E}$-valued functions in $\mathbb{D}$ whose Taylor coefficients at the origin are square summable. The shift-invariant subspaces of $H^2(\mathcal{E})$ have a description analogous to that in Beurling's theorem, the inner functions in that theorem being replaced by operator-valued analogues. Something is lost in the generalization, because the latter functions are not generally susceptible to a precise structural description like the one discussed above for scalar inner functions. (An exception is afforded by what are usually called matrix inner functions, bounded holomorphic matrix-valued functions in $\mathbb{D}$ having unitary nontangential limits almost everywhere on $\partial\mathbb{D}$. For this class of functions, and a symplectic analogue, V. P. Potapov [1955] has developed a beautiful structure theory.)

Multiple shifts play a prominent role in model theories for Hilbert space contractions. The prototypical theory of operator models is, of course, the classical spectral theorem, which in its various incarnations provides canonical models for self-adjoint operators, normal operators, one-parameter unitary groups, and commutative C^*-algebras. Model theories that go beyond the confines of the classical spectral theorem developed on several parallel fronts beginning in the 1950's. The theory originated by M. S. Livshitz [1952] and M. S. Brodskii [1956] focuses on operators that are "nearly" self-adjoint. The theories of B. Sz.-Nagy and C. Foiaş [1967] and L. de Branges and J. Rovnyak [1966, Appendix, pp. 347–392] apply to general contractions but are most effective for "nearly" unitary ones.

The spirit of these "nonclassical" model theories can be illustrated with the Volterra operator, the operator V on $L^2[0,1]$ of indefinite integration:

$$(Vf)(x) = \int_0^x f(t)\,dt \qquad \text{for } 0 \le x \le 1.$$

The adjoint of V is given by

$$(V^*f)(x) = \int_x^1 f(t)\,dt \qquad \text{for } 0 \le x \le 1,$$

from which one sees that $V + V^*$ is a positive operator of rank one, and hence that the operator $(I-V)(I+V)^{-1}$ is a contraction and a rank-one perturbation of a unitary operator.

The invariant subspaces of V were determined, by different methods, by Brodskii [1957] and W. F. Donoghue [1957]. The result is also a corollary of earlier work of S. Agmon [1949]; it says that the only invariant subspaces of V are the obvious ones, the subspaces $L^2[a,1]$ for $0 \le a \le 1$. (Here, $L^2[a,1]$ is identified with the subspace of functions in $L^2[0,1]$ that vanish off $[a,1]$.) It was eventually recognized that the Agmon–Brodskii–Donoghue result is "contained" in Beurling's theorem [Sarason 1965].

To explain the last remark we consider, for $a > 0$, the singular inner function

$$u_a(z) = \exp\left(a\left(\frac{z+1}{z-1}\right)\right),$$

and the orthogonal complement of its corresponding invariant subspace, which we denote by K_a:

$$K_a = H^2 \ominus u_a H^2.$$

We look in particular at K_1, and on K_1 we consider the operator S_1 one obtains by compressing the shift S. Thus, to apply S_1 to a function in K_1, one first multiplies the function by z and then projects the result onto K_1. (The adjoint of S_1 is the restriction of S^* to K_1.)

There is a natural isometry, involving the Cayley transform, that maps L^2 (of $\partial\mathbb{D}$) onto $L^2(\mathbb{R})$. If one follows that isometry by the Fourier–Plancherel transformation, one obtains again an isometry of L^2 onto $L^2(\mathbb{R})$. The latter isometry maps H^2 onto $L^2[0,\infty)$ and maps K_a onto $L^2[0,a]$. And it transforms the operator S_1 on K_1 to the operator $(I-V)(I+V)^{-1}$ on $L^2(0,1)$. The operator S_1 is thus a "model" of the operator $(I-V)(I+V)^{-1}$.

By Beurling's theorem, the invariant subspaces of S_1 are exactly the subspaces $uH^2 \ominus u_1H^2$ with u an inner function that divides u_1 (divides, that is, in the algebra H^∞ of bounded holomorphic functions in $\mathbb{D}$). From the structure theory for inner functions described earlier one can see that the only inner functions that divide u_1 are the functions u_a with $0 \le a \le 1$ (and their multiples by unimodular constants). On the basis of the transformation described above, one concludes that the invariant subspaces of $(I-V)(I+V)^{-1}$ are the subspaces $L^2[a,1]$ with $0 \le a \le 1$. Finally, each of the operators V and $(I-V)(I+V)^{-1}$ is easily seen to be approximable in norm by polynomials in the other, implying that these two operators have the same invariant subspaces. The Agmon–Brodskii–Donoghue result follows.

One can sum up the preceding remarks by saying that the Volterra operator, V, is "contained in" the shift operator, S. A simple and elegant observation of G. C. Rota [1960], which provides a hint of the Sz.-Nagy–Foiaş and de Branges–Rovnyak model theories, shows, startlingly, that all Hilbert space operators are "contained in" multiple shifts. Consider an operator T of spectral radius less than 1 on a Hilbert space $\mathcal{E}$. With each vector x in $\mathcal{E}$ we associate the $\mathcal{E}$-valued holomorphic function f_x given by the power series $\sum_{n=0}^{\infty} z^n T^{*n} x$. Because of the spectral condition imposed on T, the function f_x is holomorphic on $\overline{\mathbb{D}}$, so in particular it belongs to $H^2(\mathcal{E})$. The space of all such functions f_x is a subspace K of $H^2(\mathcal{E})$, invariant under the adjoint of the shift operator on $H^2(\mathcal{E})$. The map $x \to f_x$ from $\mathcal{E}$ onto K is a bounded, invertible operator that intertwines T^* with the adjoint of the shift operator. It follows that the operator T is similar to the compression of the shift operator to K. In a sense, then, multiple shifts provide replicas of all operators.

To be a bit more precise, Rota's observation provides a similarity model for every Hilbert space operator whose spectral radius is less than 1. The model space is the orthogonal complement of a shift-invariant subspace of a vector-valued H^2 space, and the model operator is the compression of the shift to the model space. The more powerful Sz.-Nagy–Foiaş and de Branges–Rovnyak theories provide unitarily equivalent models, not merely similarity models, for Hilbert space contractions. The theory of Sz.-Nagy and Foiaş springs from the subject of unitary dilations. Their model spaces include, among a wider class, the orthogonal complements of all shift-invariant subspaces of vector-valued H^2 spaces, the corresponding model operators being compressions of shifts. The model spaces of de Branges–Rovnyak are certain Hilbert spaces that live inside vector-valued H^2 spaces, not necessarily as subspaces but as contractively contained spaces, that is, spaces whose norms dominate the norms of the containing spaces.

The connection between the model theories of Sz.-Nagy–Foiaş and de Branges–Rovnyak has been explained by J. A. Ball and T. L. Kriete [1987]. Further insight was provided by N. K. Nikolskii and V. I. Vasyunin [1989], who developed what they term a coordinate-free model theory that contains, as particular cases, the Sz.-Nagy–Foiaş and de Branges–Rovnyak theories.

Interpolation

The operators S and S^* are themselves model operators in the Sz.-Nagy–Foiaş theory, and rather transparent ones. They model irreducible pure isometries and their adjoints, respectively. Next in simplicity are the compressions of S to proper S^*-invariant subspaces of H^2. For u a nonconstant inner function, let K_u denote the orthogonal complement in H^2 of the Beurling subspace uH^2, and let S_u denote the compression of S to K_u. (The action of S_u is thus multiplication by z followed by projection onto K_u.) J. W. Moeller [1962] showed that the spectrum of S_u consists of the zeros of u in $\mathbb{D}$ plus the points on $\partial\mathbb{D}$ where u has 0 as a cluster value. Moeller's paper and other considerations led the author to suspect that every operator commuting with S_u should be obtainable as the compression of an operator commuting with S. The operators of the latter kind are just the multiplication operators on H^2 induced by H^∞ functions. The result was eventually proved in a more precise form: an operator T that commutes with S_u is the compression of an operator that commutes with S and has the same norm as T [Sarason 1967]. There is a close link with two classical interpolation problems, the problems of Carathéodory–Fejér and Nevanlinna–Pick.

In the first of these problems [Carathéodory and Fejér 1911], one is given as data a finite sequence $c_0, c_1, \dots, c_{N-1}$ of complex numbers, and one wants to find a function in the unit ball of H^∞ having those numbers as its first N Taylor coefficients at the origin. To recast this as a problem about operators, let u be the inner function z^N. The functions $1, z, \dots, z^{N-1}$ form an orthonormal basis for the corresponding space K_u, and the matrix in this basis for the operator S_u has the entry 1 in each position immediately below the main diagonal and 0 elsewhere. Let $T = \sum_{j=0}^{N-1} c_j S_u^j$, so the matrix for T is lower triangular with the entry c_j in each position j steps below the main diagonal. Then T commutes with S_u, and the question of whether the Carathéodory–Fejér problem has a solution for the data $c_0, c_1, \dots, c_{N-1}$ is the same as the question of whether T is the compression of an operator commuting with S and having norm at most 1. According to the result from [Sarason 1967], T has such a compression if and only if its norm is at most 1. One recaptures in this way a solvability criterion for the Carathéodory–Fejér problem attributed by those authors to O. Toeplitz.

In the second classical interpolation problem [Nevanlinna 1919; Pick 1916], one is given as data a finite sequence $z_1, \dots, z_N$ of distinct points in $\mathbb{D}$ and a finite sequence $w_1, \dots, w_N$ of complex numbers. One wants to find a function in the unit ball of H^∞ taking the value w_j at z_j, for $j = 1, \dots, N$. For an operator reinterpretation, let u be the finite Blaschke product with zero sequence $z_1, \dots, z_N$. The space K_u is spanned by the kernel functions for the points $z_1, \dots, z_N$, the functions $k_j(z) = (1 - \bar{z}_j z)^{-1}$, where $j = 1, \dots, N$. The distinctive property of k_j is that the linear functional it induces on H^2 is the functional of evaluation at z_j. From this one sees that $S^* k_j = \bar{z}_j k_j$, so the functions $k_1, \dots, k_N$ form an eigenbasis for S_u^*. Let the operator T on K_u be defined by $T^* k_j = \bar{w}_j k_j$.

Then T commutes with S_u, and the question of whether the Nevanlinna–Pick problem is solvable for the given data is the same as the question of whether T is the compression of an operator that commutes with S and has norm at most 1. By the result from [Sarason 1967], the latter happens if and only if the norm of T is at most 1, which is easily seen to coincide with Pick's solvability criterion, namely, the positive semidefiniteness of the matrix

$$\left(\frac{1-w_j\bar{w}_k}{1-z_j\bar{z}_k}\right)_{j,k=1}^{N}.$$

Subsequently, Sz.-Nagy and Foiaş established their famous commutant-lifting theorem [1968; 1967], according to which the result from [Sarason 1967] is a special case of a general property of unitary dilations. The commutant-lifting theorem provides an operator approach to a variety of interpolation problems. The books [Rosenblum and Rovnyak 1985; Foiaş and Frazho 1990], are good sources for this material. H. Dym's review of the latter book [Dym 1994] is also recommended.

The commutant-lifting approach is just one of several operator approaches to interpolation problems. In the same year that the commutant-lifting theorem appeared, V. M. Adamyan, D. Z. Arov and M. G. Krein published the first two of a remarkable series of papers on the Nehari interpolation problem [Adamyan et al. 1968b; 1968a]. In the Nehari problem one is given as data a sequence $(c_n)_{n=1}^{\infty}$ of complex numbers, and one wants to find a function f in the unit ball of L^∞ (on the unit circle) having these numbers as its negatively indexed Fourier coefficients (i.e., $\hat{f}(-n)=c_n$ for $n=1,2,\ldots$). Z. Nehari [1957] proved that the problem is solvable if and only if the Hankel matrix $(c_{j+k+1})_{j,k=0}^{\infty}$ has norm at most 1 as an operator on l^2. Using a method akin to the operator approach to the Hamburger moment problem, Adamyan, Arov and Krein proved that finding a solution f, in case one exists, is tantamount to finding a unitary extension of a certain isometric operator constructed from the data. What is more, the family of all such solutions is in one-to-one correspondence with the family of all such unitary extensions (satisfying a minimality requirement), a connection that enabled them to derive a linear-fractional parameterization of the set of all solutions in case the problem is indeterminate. For the indeterminate Nevanlinna–Pick problem, a linear-fractional parameterization of the solution set was found by Nevanlinna [1919] on the basis of the Schur algorithm, a technique invented by I. Schur [1917] in connection with the Carathéodory–Fejér problem. Nevanlinna's parameterization, and the corresponding one implicit in Schur's paper, can be deduced from the one of Adamyan, Arov and Krein, because one can show that the Nehari problem embraces the Carathéodory–Fejér and Nevanlinna–Pick problems.

There is a close connection between the commutant-lifting theorem and the work of Nehari and Adamyan, Arov and Krein. Nehari's theorem is in fact a corollary of the commutant-lifting theorem; the details can be found, for example, in [Sarason 1991]. In the other direction, it was recognized by D. N. Clark (unpublished notes) and N. K. Nikolskii [1986] that the theorem from [Sarason 1967] can be deduced very simply from Nehari's theorem, and more recently R. Arocena [1989] has shown how to give a proof of the commutant-lifting theorem using the methods of Adamyan, Arov and Krein. Further discussion can be found in [Sarason 1987; 1991].

There follows a brief description of some other approaches to interpolation problems.

• The Abstract Interpolation Problem of V. E. Katsnelson, A. Ya. Kheifets and P. M. Yuditskii [Katsnelson et al. 1987; Kheifets and Yuditskii 1994] is based on the approach of V. P. Potapov and coworkers to problems of Nevanlinna–Pick type [Kovalishina and Potapov 1974; Kovalishina 1974; 1983]. It abstracts the key elements of Potapov's theory to an operator setting and yields, upon specialization, a wide variety of classical problems. The model spaces of de Branges–Rovnyak play an important role in this approach. As was the case with the Adamyan–Arov–Krein treatment of the Nehari problem, the solutions of the Abstract Interpolation Problem correspond to the unitary extensions of a certain isometric operator. There is a unified derivation of the linear-fractional parameterizations of the solution sets of indeterminate problems.

• The approach favored by H. Dym [1989] emphasizes reproducing kernel Hilbert spaces, especially certain de Branges–Rovnyak spaces, and J-inner matrix functions. (The J here refers to a signature matrix, a square, self-adjoint, unitary matrix. A meromorphic matrix function in $\mathbb{D}$, with values of the same size as J, is called J-inner if it is J-contractive at each point of $\mathbb{D}$ where it is holomorphic, and its boundary function is J-unitary almost everywhere on $\partial\mathbb{D}$. These are the symplectic analogues of inner functions that, as mentioned earlier, have been analyzed by Potapov [1955].)

• J. A. Ball and J. W. Helton [1983] have developed a Krein space approach to interpolation problems. In their approach an interpolation problem, rather than being reinterpreted as an operator extension problem, is reinterpreted as a subspace extension problem in a suitable Krein space. Shift-invariant subspaces of vector H^2 spaces that are endowed with a Krein space structure arise. One of the key results is a Beurling-type theorem for such subspaces. A treatment of the Nehari problem using this method can be found in [Sarason 1987].

• J. Agler [1989] has, in a sense, axiomatized the Nevanlinna–Pick problem and obtained the analogue of Pick's criterion in two new contexts, interpolation by multipliers of the Dirichlet space (the space of holomorphic functions in $\mathbb{D}$ whose derivatives are square integrable with respect to area), and interpolation by bounded holomorphic functions in the bidisk.

• M. Cotlar and C. Sadosky [1994] have used their theory of Hankel forms to attack problems of Nevanlinna–Pick and Nehari type in the polydisk.

• B. Cole, K. Lewis and J. Wermer have attacked problems of Nevanlinna–Pick type from the perspective of uniform algebras [Cole et al. 1992].

The foregoing list is but a partial sample of the enormous activity surrounding interpolation problems.

Systems Theory

In the early 1970's J. W. Helton became aware that there is a large overlap between the mathematics of linear systems theory and the operator theory that had grown around dilation theory and interpolation problems. In an April 17, 1972 letter to the author, he wrote: "I've spent the year learning engineering systems which at some levels is almost straight operator theory. Some of the best functional analysis (Krein, Livsic) has come from engineering institutes and I'm beginning to see why. Such collaboration does not exist in this country.... The [Sz.-]Nagy–Foiaş canonical model theory is precisely a study of infinite dimensional discrete time systems which lose and gain no energy."

Helton embarked upon a program to bridge the chasm between operator theorists and engineers in the United States. The result has been an enrichment of both mathematics and engineering. The systems theory viewpoint now permeates a large part of operator theory. On the engineering side, a new subject, H^∞ control, has sprung up [Francis 1987].

Bergman Spaces and the Bergman Shift

The mathematics discussed above flows, in large part, from Beurling's theorem via its generalization to vector H^2 spaces, in other words, to multiple shifts. Another natural direction for exploration unfolds when one replaces the shift, not by a multiple version of itself, but by its analogue (multiplication by z) on a holomorphic space of scalar functions other than H^2. There are countless possibilities for this other space; one that has turned out to be especially interesting is the Bergman space.

What was just referred to as "the" Bergman space is really just the most immediate member of a large family of spaces. Given a bounded domain in the complex plane and a positive number p, the Bergman space with exponent p for the domain consists of the holomorphic functions in the domain that are p-th power integrable with respect to area. These spaces are named for S. Bergman because the ones with exponent 2, which are Hilbert spaces, played a fundamental role in much of his work [Bergman 1970]. In the unit disk, the Bergman space with exponent p is denoted by A^p (or B^p, or L^p_a), and it is given the norm (or "norm," if $p < 1$) inherited from L^p of normalized area measure on the disk.

The powers of z form an orthogonal basis for the Hilbert space A^2, the norm of z^n being $1/\sqrt{n+1}$. Thus, a holomorphic function in $\mathbb{D}$ belongs to A^2 if and only if its Taylor coefficients at the origin are square summable when weighted