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Introduction

Relativistic numerical hydrodynamics is currently a field of intense interest. On the one hand, the development of next-generation laser interferometric and cryogenic gravity wave detectors is opening a new window of astronomy, one which will peer into a world of multidimensional rapidly varying matter and gravity fields such as occur in and around neutron stars, black holes, supernovae, compact binary systems, dense clusters, collapsing stars, the early universe, etc. At the same time, X-ray and γ-ray observatories are providing (or will soon provide) a wealth of data on the evolution of matter in and around X-ray and γ-ray emitting compact objects such as accreting black holes and neutron stars. Such systems can only be realistically analyzed by a detailed numerical study of the spacetime and matter fields.

A quantitative understanding of these systems as well as a host of other astrophysical phenomena such as stellar collapse leading to supernovae, the evolution of massive stars, and the origin of γ-ray bursts, the origin and evolution of relativistic jets, all require multidimensional complex relativistic numerical simulations in three spatial dimensions. Since analytic and post-Newtonian methods are only applicable for systems of special symmetry and/or relatively weak fields, numerical relativistic hydrodynamics is the only viable method to model such highly dynamical asymmetrical strong field systems.

The technology for observing such energetic astrophysical phenomena has developed in concert with the development of high speed computing. Hence, it is perhaps no accident that the requirement for next-generation multi-dimensional relativistic hydrodynamics modeling is occurring at a time when computers are just now approaching the speed and memory capability needed to explore such systems. For these reasons, it is expected that there will be much research in relativistic numerical hydrodynamics calculations in the coming years, hence the need for a book reviewing the development of the subject.

The textbooks from which most of us learn general relativity usually emphasize a number of analytic solutions of some special cases, like that of an isotropic Schwarzschild or Friedmann metric. Indeed, one is hard pressed to think of a problem in relativity which can still be addressed with paper and pencil. The remaining real-world applications in astrophysics and cosmology cannot be seriously studied analytically, nor can one ignore the hydrodynamic evolution of the matter fields. Such systems must be studied numerically. Indeed, the solution of numerical problems often requires one to abandon some or all aspects of Newtonian or even post-Newtonian intuition. Our goal here will be to provide an overview of the computational framework in which such calculations have been done, along with illustrative applications to real physical systems.

This book does not, however, attempt to give a comprehensive overview of how to do numerical relativistic hydrodynamics calculations. It is rather a compilation of those projects with which one or both of the authors have had some involvement. An attempt at a comprehensive overview of a field in which there have been so many significant contributors would be difficult. Hence, although we shall refer here to a number of other works in the field, this text will for the most part only summarize the contributions of the authors and collaborators. These are the works with which we are most familiar. Nevertheless, in the process of reading this text, it is hoped that the reader will gain some understanding of the development of relativistic hydrodynamics which has occurred over the past 30 years.

In what follows we will assume that the reader has some familiarity with basic concepts in special and general relativity. We only provide enough introductory material so that the relativistic field and matter equations can be introduced in a context which is most easily applied to numerical problems, and not in the way they might be introduced in an introductory text in either relativity or hydrodynamics alone.

1.1 Notation and convention

As with any other intensely mathematical subject, a text on numerical relativity should contain a concise summary of notation and convention in one location. Hence, we begin with an overview of the notation and conventions which we have attempted to maintain throughout the book. By and large, these are the conventions widely adopted in the field, and as such, comprise useful introductory material.

In what follows we use the usual convention of Greek indices to denote components of four-dimensional spacetime ($\mu = 0, 1, 2, 3$). When referring to a specific coordinate system they will be identified according to normal convention (e.g. $\mu = t, x, y, z$ for Cartesian coordinates). We use Latin characters, $i, j, k, \ldots$ to denote spatial indices. Partial differentiation will be denoted in both the explicit and abbreviated form, e.g.

$$\frac{\partial}{\partial x^\mu} = \partial_\mu. \tag{1.1}$$

Partial differentiation along the time coordinate will also frequently be denoted by the familiar "dot" notation,

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial x^0} = \dot{A}. \tag{1.2}$$

We will also make use of geometrized units, $c = G = 1$. For convenience, Table 1.1 gives conversions from cgs units to geometrized units for various parameters in use in this text.

1.2 General relativity

A brief summary of general relativity is a necessary starting point for introducing concepts and notation to be encountered in subsequent chapters. General relativity derives from the principle of equivalence which asserts that at every spacetime point we can choose a coordinate system such that the laws of physics have the same form as they would in the absence of a gravitational field. This principle has led to the Einstein field equations which relate the curvature of spacetime to the distribution of mass–energy,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.3}$$

where $T_{\mu\nu}$ is the energy momentum (or stress energy) tensor.

The Einstein tensor $G_{\mu\nu}$ can be written in terms of the Ricci tensor $R_{\mu\nu}$, metric tensor $g_{\mu\nu}$, and Ricci scalar R,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{1.4}$$

where the Ricci tensor is a contraction of the Reimann tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \tag{1.5}$$

and

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.6}$$

Table 1.1 Conversion from cgs units to geometrized units

Quantity	cgs units	Geometrized units
Length	l (cm)	(cm)
Time	t (s)	$ct = 2.99792458 \times 10^{10} \times t$ (cm)
Velocity	V(cm s^{-1})	$v/c = v/2.99792458 \times 10^{10}$ (dimensionless)
Mass	m (g)	$Gm/c^2 = 7.42426 \times 10^{-29} \times m$ (cm)
Energy	e (erg)	$Ge/c^4 = 8.26060 \times 10^{-50} \times e$ (cm)
Internal energy	ϵ (erg g^{-1})	$\epsilon/c^2 = 1.11265 \times 10^{-21} \times \epsilon$ (dimensionless)
Mass density	ρ (g cm^{-3})	$G\rho/c^2 = 7.42426 \times 10^{-29} \times \rho$ (cm^{-2})
Energy density	$\rho\epsilon$ (erg cm^{-3})	$G\rho\epsilon/c^4 = 8.26060 \times 10^{-50} \times \rho\epsilon$ (cm^{-2})
Pressure	P (dyn cm^{-2})	$GP/c^4 = 8.26060 \times 10^{-50} \times P$ (cm^{-2})
Temperature	T (K)	$GkT/c^4 = 8.26060 \times 10^{-50} \times kT$ (cm)
Entropy	S/k (dimensionless)	S/k (dimensionless)
Angular momentum	J (g cm^2 s^{-1})	$GJ/c^3 = 2.47647 \times 10^{-39} J$ (cm^2)
Angular frequency	ω (rad s^{-1})	$\omega/c = 3.33564 \times 10^{-11} \omega$ (cm^{-1})
Luminosity	L (erg s^{-1})	$GL/c^5 = 2.75544 \times 10^{-60} L$ (dimensionless)
Solar mass	$M_\odot = 1.989 \times 10^{33}$ g	$GM_\odot/c^2 = 1.477$ km
Solar luminosity	$L_\odot = 3.826 \times 10^{33}$ erg s^{-1}	$GL_\odot/c^5 = 1.054 \times 10^{-26}$ (dimensionless)
Nuclear density	$\rho_N = 2.67 \times 10^{14}$ g cm^{-3}	$G\rho_N/c^2 = 1.98 \times 10^{-14}$ (cm^{-2})

Here, the Riemann–Christoffel curvature tensor $R^\sigma{}_{\mu\kappa\nu}$ is

$$R^\sigma{}_{\mu\kappa\nu} = \partial_\kappa \Gamma^\sigma{}_{\mu\nu} - \partial_\nu \Gamma^\sigma{}_{\mu\kappa} + \Gamma^\sigma{}_{\eta\kappa}\Gamma^\eta{}_{\mu\nu} - \Gamma^\sigma{}_{\eta\nu}\Gamma^\eta{}_{\mu\kappa}, \tag{1.7}$$

where the Christoffel symbols, $\Gamma^\alpha{}_{\mu\nu}$, relate directly to the metric tensor. For the usual case of a coordinate system in which the basis vectors commute we have,

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left\{ \frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right\}. \tag{1.8}$$

1.2.1 Metric tensor

From Eqs. (1.4)–(1.8) we see that the geometry of spacetime is specified once the metric $g_{\mu\nu}$ and its derivatives are given. The generalization from special relativity to general relativity is then simply the generalization from a Euclidean flat space metric tensor to a curved space metric. As in special relativity, the infinitesimal proper interval between two events in spacetime is denoted

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.9}$$

Now however, $g_{\mu\nu}$ can no longer be described by a simple Minkowski metric, but instead involves curvature. In this book we will consistently use the Misner, Thorne and Wheeler [13] sign conventions whereby the Einstein equation and the Riemann tensor have a positive sign as written above and $g_{\mu\nu}$ is *space like*, e.g. in flat space,

$$g^{flat}_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.10}$$

1.2.2 Energy momentum tensor

The other side of the Einstein equation (1.3), and the part of most interest for relativistic hydrodynamics, is of course the stress energy tensor, $T_{\mu\nu}$. In a frame of reference in which a perfect fluid is in motion with respect to an observer, the energy momentum tensor is written most generally as

$$T_{\mu\nu} = (\rho + \rho\epsilon + P) U_\mu U_\nu + P g_{\mu\nu}, \tag{1.11}$$

where ρ is the local baryon rest-mass energy density. ρ is related to the baryon number density n_b

$$\rho = m_0 n_b, \tag{1.12}$$

where m_0 is the baryon rest mass appropriate to the matter composition. The quantity ϵ contains all information about the net internal energy per unit mass of the baryons. It can be less than zero, for example for a nondegenerate gas of bound nuclei. The quantity P in Eq. (1.11) is the pressure, and U_μ is the covariant four-velocity. In a reference frame that is at rest and locally Lorentzian, the stress energy tensor for an isotropic perfect fluid can be written in a familiar form,

$$T_{\mu\nu} = \begin{pmatrix} \rho(1+\epsilon) & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \tag{1.13}$$

However, this form is not particularly useful for hydrodynamic simulations with nontrivial fluid motion. In what follows we will deal almost exclusively with Eq. (1.11), correcting for imperfect fluids and other fields (e.g. electromagnetic) where appropriate.

1.2.3 Covariant differentiation

Equations of motion in general relativity require the introduction of covariant differentiation. We use the notation $A^\mu{}_{;\nu}$ or $D_\nu A^\mu$ to denote covariant differentiation of a contravariant vector A^μ,

$$D_\nu A^\mu = A^\mu{}_{;\nu} = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^\mu{}_{\nu\lambda} A^\lambda. \tag{1.14}$$

Similarly, the covariant derivative of a covariant vector is

$$A_{\mu;\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\lambda{}_{\mu\nu} A_\lambda. \tag{1.15}$$

In what follows we will also introduce covariant differentiation in the ADM three-space (cf. Section 1.3). This we denote:

$$D_i A^j = \frac{\partial A_i}{\partial x^j} + \Gamma^i{}_{jk} A^k, \tag{1.16}$$

where $\Gamma^i{}_{jk}$ now denotes connection coefficients for the three-dimensional ADM hypersurface [22].

$$\Gamma^i{}_{jk} = \frac{1}{2}\gamma^{il}\left\{\frac{\partial \gamma_{lj}}{\partial x^k} + \frac{\partial \gamma_{lk}}{\partial x^j} - \frac{\partial \gamma_{jk}}{\partial x^l}\right\} + \frac{1}{2}\left(\gamma^{il}\gamma_{km}C^m{}_{lj} + \gamma^{il}\gamma_{jm}C^m{}_{lk} - C^i{}_{jk}\right), \tag{1.17}$$

where the $C^i{}_{jk}$ enter when the basis vectors of the three-space (denoted e_i) do not commute, $[e_i, e_j] = e_k C^k{}_{ij} \neq 0$. For most applications, the simplifications embodied in commuting coordinates are desirable so that we can take $C^k{}_{ij} = 0$ and the three-space connection coefficient simply becomes the usual Christoffel symbol defined in terms of γ_{ij}.

The covariant derivative of a scalar α is just the ordinary gradient

$$\alpha_{;\mu} = \frac{\partial \alpha}{\partial x^\mu}. \tag{1.18}$$

The generalization of covariant differentiation to tensors of higher rank is straightforward. For each contravariant index μ a term with $\Gamma^\mu{}_{\nu\lambda}$ times the tensor is added, but with μ in the tensor replaced by λ. For each covariant index ν one subtracts a term with $\Gamma^\kappa{}_{\nu\lambda}$ times the tensor with ν replaced with κ. For example,

$$D_\rho T^\mu{}_\nu = T^\mu{}_{\nu;\rho} = \frac{\partial T^\mu{}_\nu}{\partial x^\rho} + \Gamma^\mu{}_{\rho\lambda} T^\lambda{}_\nu - \Gamma^\kappa{}_{\nu\rho} T^\mu{}_\kappa. \tag{1.19}$$

A particularly useful operation when deriving the hydrodynamic equations of motion is the covariant divergence. For a vector this simplifies to

$$V^\mu{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{g} V^\mu\right). \tag{1.20}$$

For a tensor it simplifies to

$$T^{\mu\nu}{}_{;\mu} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left(\sqrt{g} T^{\mu\nu}\right) + \Gamma^\lambda{}_{\mu\nu} T^{\mu\nu}, \tag{1.21}$$

where g is the negative of the determinant of the metric tensor

$$g \equiv -\det(g_{\mu\nu}). \tag{1.22}$$

The above relations extend trivially to covariant differentiation in the ADM three-space by simply writing them in terms of spatial indices. In what follows, we will usually denote the determinant of the three-metric by

$$\gamma^2 \equiv \det(\gamma_{ij}). \tag{1.23}$$

1.2.4 Bianchi identities

The Riemann curvature tensor obeys some special symmetries known as the Bianchi identities,

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0. \tag{1.24}$$

The contracted form of this equation implies [19] that the covariant divergence of the Einstein tensor also vanishes

$$G^{\mu\nu}{}_{;\nu} = 0. \tag{1.25}$$

This, together with the Einstein equation (1.3), immediately implies,

$$T^{\mu\nu}{}_{;\nu} = 0, \tag{1.26}$$

from which the fluid equations of motion will be derived in subsequent chapters.

In principle this is all we need to describe any system. The problem, however, is that the Einstein equations as they stand are too general. What is needed is a reduction of the Einstein equations to a form suitable for evolving in time. This we now describe.

1.3 (3+1) or ADM formalism

In general relativity and special relativity the distinction between spatial three dimensions and the time dimension becomes obscure. Indeed, it is an intriguing and beautiful aspect of relativity that what one means by time and space depends upon which observer is making the determination. However, in spite of this ambiguity, one wishes to have a method to describe a system which is "evolving in time," as this is the way we perceive events in Nature.

A way to approach this problem which nearly aligns with Newtonian intuition is to construct successive "snapshots" or "time slices" of the spatial (curved) three-space geometry along a sequence of steps in a time-like coordinate t. That is, spacetime is sliced (or foliated) into a one-parameter family of hypersurfaces separated by differential displacements in coordinate time t.

Indeed, there are many ways in which one could follow a time-like dimension and watch events unfold in spacetime. However, the one which most nearly aligns with Newtonian intuition (and one which is usually amenable to numerical methods) is the ADM (after its inventors Arnowitt, Deser and Misner [3]) or $(3 + 1)$ formalism [13].

In this approach, the time evolution of the metric is expressed as first-order time derivatives, while $G_{\mu\nu}$ contains second-order time derivatives. The time-like coordinate is chosen along a normal to the space-like hypersurfaces. Figure 1.1 shows a two-dimensional spacetime depiction of one way to do this division. That is, space and time are placed on separate footings by first specifying the proper time interval $\vec{d\tau}$ between the lower and upper hypersurfaces along the direction of the normal $\vec{n}$ to

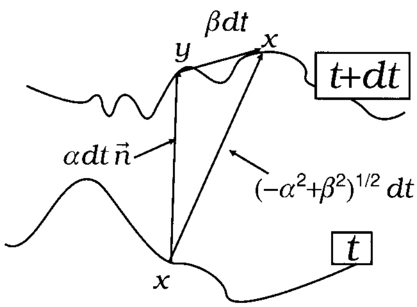

Fig. 1.1. Schematic depiction of the ADM metric.

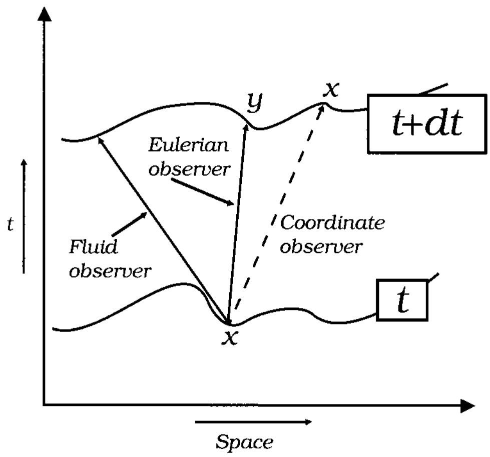

Fig. 1.2. Schematic depiction of observer frames in the ADM metric.

the spatial hypersurfaces,

$$\vec{d\tau} = dt\vec{n}. \tag{1.27}$$

We call an observer in this frame, the *Eulerian observer* as depicted in Figure 1.2.

1.3.1 Eulerian observer

An Eulerian observer moves through spacetime in a direction orthogonal to all spacetime vectors confined to the spatial hypersurfaces. This frame does not necessarily move along a force-free trajectory or geodesic in spacetime. Since the Eulerian frame is defined independently of the coordinates chosen, it is a good frame in which to measure physical quantities such as fluid velocity, neutrino energy, etc. The Einstein equations are solved in this Eulerian frame by projecting them onto the

the unit normal to the time slices n_μ. To Eulerian observers, the perfect-fluid ADM stress energy tensor then appears to be an imperfect fluid with the following components:

$$T_{\mu\nu} = dn_\mu n_\nu + s_\mu n_\nu + n_\mu s_\nu + S_{\mu\nu}, \tag{1.28}$$

where

$$\rho_H \equiv n^\mu n^\nu T_{\mu\nu}, \tag{1.29}$$

$$s_\mu \equiv n^\delta h_\mu^\beta T_{\beta\delta}, \tag{1.30}$$

$$S_{\mu\nu} = h_\mu^\delta h_\nu^\beta T_{\delta\beta}. \tag{1.31}$$

Here, h_μ^ν is a projection operator onto the three-slices and is written in terms of the time-like unit normal (Eqs. (1.44), (1.45)),

$$h_\mu^\nu \equiv \delta_\mu^\nu + n_\mu n^\nu. \tag{1.32}$$

The quantity ρ_H is called the *Hamiltonian density*. It is an ADM matter density related to hydrodynamic variables as

$$\rho_H = \rho h W^2 - P, \tag{1.33}$$

where ρ is the baryon rest mass energy, Eq. (1.12). The quantity h is called the *specific enthalpy*. Written in terms of the internal energy per gram of material ϵ and the pressure P it is

$$h \equiv 1 + \epsilon + P/\rho. \tag{1.34}$$

The quantity $W \equiv \alpha U^t$ is a generalized Lorentz factor described below.

The quantity s_μ is the *ADM momentum density*. In covariant form its spatial components are equivalent to the spatial components of the relativistic four-momentum density $S_\mu \equiv \rho_h W U_\mu$,

$$s_i = S_i = \rho h W U_i. \tag{1.35}$$

The space–space component of $S_{\mu\nu}$ is called the *spatial stress*. For a perfect-fluid stress energy tensor, the spatial stress reduces to

$$S_{ij} = P\gamma_{ij} + \rho h U_i U_j = P\gamma_{ij} + \frac{S_i S_j}{\rho h W^2}. \tag{1.36}$$