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Introduction

FOUNDATIONS have acquired a bad name amongst mathematicians, because of the reductionist claim analogous to saying that the atomic chemistry of carbon, hydrogen, oxygen and nitrogen is enough to understand biology. Worse than this: whereas these elements are known with no question to be fundamental to life, the membership relation and the Sheffer stroke have no similar status in mathematics.

Our subject should be concerned with the basic idioms of argument and construction in mathematics and programming, and seek to explain these (as fundamental physics does) in terms of more general and basic phenomena. This is "discrete math for grown-ups".

A moderate form of the logicist thesis is established in the tradition from Weierstrass to Bourbaki, that mathematical treatments consist of the manipulation of assertions built using $\wedge$, $\vee$, $\Rightarrow$, $\forall$ and $\exists$. We shall show how the way in which *mathematicians* (and programmers) — rather than logicians — conduct such discussions really does correspond to a certain semi-formal system of proof in (intuitionistic) predicate calculus. The working mathematician who is aware of this correspondence will be more likely to make valid arguments, that others are able to follow. Automated deduction is still in its infancy, but such awareness may also be expected to help with computer-assisted construction, verification and dissemination of proofs.

One of the more absurd claims of extreme logicism was the reduction of the natural numbers to the predicate calculus. Now we have a richer view of what constitutes logic, based on a powerful *analogy between types and propositions*. In classical logic, as in classical physics, particles enact a logical script, but neither they nor the stage on which they perform are permanently altered by the experience. In the modern view, matter and its activity are created together, and are interchangeable (the observer also affects the experiment by the strength of the meta-logic).

This analogy, which also makes algebra, induction and recursion part of logic, is a structural part of this book, in that we always treat the simpler propositional or order-theoretic version of a phenomenon as well as the type or categorical form.

Besides this and the classical symmetry between $\wedge$ and $\vee$ and between $\forall$ and $\exists$, the modern rules of logic exhibit one between introduction (proof, element) and elimination (consequence, function). These rules are part of an even wider picture, being examples of *adjunctions*.

This suggests a new understanding of foundations apart from the mere codification of mathematics in logical scripture. When the connectives and quantifiers have been characterised as (universal) properties of mathematical structures, we can ask what *other* structures admit these properties. Doing this for coproducts in particular reveals rather a lot of the elementary theory of algebra and topology. We also look for function spaces and universal quantifiers among topological spaces.

A large part of this book is category theory, but that is because for many applications this seems to me to be the most efficient heuristic tool for investigating structure, and comparing it in different examples. Plainly we all write mathematics in a symbolic fashion, so there is a need for fluent translations that render symbols and diagrams as part of the same language. However, it is not enough to see recursion as an example of the adjoint functor theorem, or the propositions-as-types analogy as a reflection of bicategories. We must also *contrast* the examples and give a *full* categorical account of symbolic notions like structural recursion.

You should not regard this book as yet another foundational prescription. I have deliberately not given any special status to any particular formal system, whether ancient or modern, because I regard them as the vehicles of meaning, not its cargo. I actually believe the (moderate) logicist thesis less than most mathematicians do. This book is not intended to be synthetic, but analytic — to ask what mathematics requires of logic, with a view to starting again from scratch [Tay98].

Advice to the reader. Technical books are never written and seldom read sequentially. Of course you have to know what a category is to tackle Chapter V, but otherwise it is supposed to be possible to read any of at least the first six chapters on the basis of general mathematical experience alone; the people listed below were given *individual* chapters to read partly in order to ensure this. There is more continuity between sections, but again, if you get stuck, *move on to the next one*, as secondary material is included at the end of some sections (and subsections). The book is thoroughly indexed and cross-referenced to take you as quickly as possible to specific topics; when you have found what you want, the cross-references should be ignored.

The occasional anecdotes are not meant to be authoritative history. They are there to remind us that *mathematics is a human activity*, which is always done in some social and historical context, and to encourage the reader to trace its roots. The dates emphasise quite how late logic arrived on the mathematical scene. The footnotes are for colleagues, not general readers, and there are theses to be written *à propos* of the exercises.

The first three chapters should be accessible to final year undergraduates in mathematics and informatics; lecturers will be able to select appropriate sections themselves, but should warn students about parenthetical material. Most of the book is addressed to graduate students; Section 6.4 and the last two chapters are research material. I hope, however, that every reader will find interesting topics throughout the book.

Chapters IV, V and VII provide a course on category theory. Chapter III is valuable as a prelude since it contains many of the results (the adjoint functor theorem, for example) in much simpler form.

Sections 1.1–5 (not necessarily in that order), 2.3, 2.4, 2.7, 2.8, 3.1–5, 4.1–5 and 4.7 provide a course on the semantics of the λ-calculus.

For imperative languages, take Sections 1.4, 1.5, 4.1–6, 5.3, 5.5 and 6.4.

An advanced course on type theory would use Chapter IV as a basis, followed by Chapters V and IX with Sections 7.6, 7.7 to give semantic and syntactic points of view on similar issues.

Chapter VI and Section 9.6 discuss topics in symbolic logic using the methods category theory.

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When I began this book in 1991, I was a member of a lively research group led by Samson Abramsky at Imperial College. Those who were there, including Roy Crole, Simon Gay, Radha Jagadeesan, Achim Jung, Yves Lafont, François Lamarche, Ian Mackie, Raja Nagarajan, Luke Ong, Duško Pavlović, Christian Retoré, Leopoldo Román, Mark Ryan, Mike Smyth and Steve Vickers, provided much stimulation. But in 1996 the management saw fit first to deprive me of my office, and later to withhold my EPSRC salary for not being in that office. I am deeply indebted to everyone at Queen Mary and Westfield College for supporting me at this distressing time: besides being a friendlier place all round, QMW has a much healthier working environment. I have since learned a lot from Richard Bornat, Peter Burton, Keith Clarke, Adam Eppendahl, Peter Landin, Peter O'Hearn, David Pym, Edmund Robinson and Graham White.

Jim Lambek was the first of my senior colleagues to express appreciation of this work. At many times when I might otherwise have given up, Carolyn Brown, Adam Eppendahl, Pino Rosolini and Graham White told me repeatedly that it was worth the effort.

Since this is my first and a very personal book, I would also like to record my appreciation of those who have taught me and encouraged my career in mathematics, beginning with my parents, Brenda and Ced(ric) Taylor. Ruth Horner of Stoke Poges county primary school, Buckinghamshire; Christian Puritz, Bert Scott, Doris Wilson and the late Henry Talbot at the Royal Grammar School, High Wycombe; Béla Bollobás, Andrew Casson and Pelham Wilson when I was an undergraduate at Trinity.

Typography. I composed this book using Emacs and typeset it all myself in TEX: I cannot conceive of doing research without these two programs, and all of the software that I have used is public domain (for a long time Mark Dawson kept this going for me). The commutative diagram, proof tree, proof box and design macros are my own. I would like to thank Vera Brice and Leslie Robinson of the London College of Printing for their suggestions and an interesting course in book design. Without Peter Jackson's eagle eye and Roger Astley's patient guidance at Cambridge University Press, however, there would be far more errors and idiosyncracies than you see here now.

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