Vorticity and Incompressible Flow

This book is a comprehensive introduction to the mathematical theory of vorticity and incompressible flow ranging from elementary introductory material to current research topics. Although the contents center on mathematical theory, many parts of the book showcase the interactions among rigorous mathematical theory, numerical, asymptotic, and qualitative simplified modeling, and physical phenomena. The first half forms an introductory graduate course on vorticity and incompressible flow. The second half comprises a modern applied mathematics graduate course on the weak solution theory for incompressible flow.

Andrew J. Majda is the Samuel Morse Professor of Arts and Sciences at the Courant Institute of Mathematical Sciences of New York University. He is a member of the National Academy of Sciences and has received numerous honors and awards including the National Academy of Science Prize in Applied Mathematics, the John von Neumann Prize of the American Mathematical Society and an honorary Ph.D. degree from Purdue University. Majda is well known for both his theoretical contributions to partial differential equations and his applied contributions to diverse areas besides incompressible flow such as scattering theory, shock waves, combustion, vortex motion and turbulent diffusion. His current applied research interests are centered around Atmosphere/Ocean science.

Andrea L. Bertozzi is Professor of Mathematics and Physics at Duke University. She has received several honors including a Sloan Research Fellowship (1995) and the Presidential Early Career Award for Scientists and Engineers (PECASE). Her research accomplishments in addition to incompressible flow include both theoretical and applied contributions to the understanding of thin liquid films and moving contact lines.

Cambridge Texts in Applied Mathematics

Maximum and Minimum Principles
M. J. SEWELL

Solitons
P. G. DRAZIN AND R. S. JOHNSON

The Kinematics of Mixing
J. M. OTTINO

Introduction to Numerical Linear Algebra and Optimisation
PHILIPPE G. CIARLET

Integral Equations
DAVID PORTER AND DAVID S. G. STIRLING

Perturbation Methods
E. J. HINCH

The Thermomechanics of Plasticity and Fracture
GERARD A. MAUGIN

Boundary Integral and Singularity Methods for Linearized Viscous Flow
C. POZRIKIDIS

Nonlinear Wave Processes in Acoustics
K. NAUGOLNYKH AND L. OSTROVSKY

Nonlinear Systems
P. G. DRAZIN

Stability, Instability and Chaos
PAUL GLENDINNING

Applied Analysis of the Navier–Stokes Equations
C. R. DOERING AND J. D. GIBBON

Viscous Flow
H. OCKENDON AND J. R. OCKENDON

Scaling, Self-Similarity, and Intermediate Asymptotics
G. I. BARENBLATT

A First Course in the Numerical Analysis of Differential Equations
ARIEH ISERLES

Complex Variables: Introduction and Applications
MARK J. ABLOWITZ AND ATHANASSIOS S. FOKAS

Mathematical Models in the Applied Sciences
A. C. FOWLER

Thinking About Ordinary Differential Equations
ROBERT E. O'MALLEY

A Modern Introduction to the Mathematical Theory of Water Waves
R. S. JOHNSON

Rarefied Gas Dynamics
CARLO CERCIGNANI

Symmetry Methods for Differential Equations
PETER E. HYDON

High Speed Flow
C. J. CHAPMAN

Wave Motion
J. BILLINGHAM AND A. C. KING

An Introduction to Magnetohydrodynamics
P. A. DAVIDSON

Linear Elastic Waves
JOHN G. HARRIS

Introduction to Symmetry Analysis
BRIAN J. CANTWELL

Vorticity and Incompressible Flow

ANDREW J. MAJDA
New York University

ANDREA L. BERTOZZI
Duke University

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK
40 West 20th Street, New York, NY 10011-4211, USA
10 Stamford Road, Oakleigh, VIC 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain
Dock House, The Waterfront, Cape Town 8001, South Africa

http://www.cambridge.org

First published 2002

Printed in the United Kingdom at the University Press, Cambridge

Typeface Times Roman 10/13 pt. *System* LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library.

Library of Congress Cataloging in Publication Data
Majda, Andrew, 1949–
Vorticity and incompressible flow / Andrew J. Majda, Andrea L. Bertozzi.
p. cm. – (Cambridge texts in applied mathematics ; 27)
Includes bibliographical references and index.
ISBN 0-521-63057-6 (hb) – ISBN 0-521-63948-4 (pb)
1. Vortex-motion. 2. Non-Newtonian fluids. I. Bertozzi, Andrea L. II. Title. III. Series.
QA925 .M35 2001
532′.059 – dc21 00-046776

ISBN 0 521 63057 6 hardback
ISBN 0 521 63948 4 paperback

Contents

Preface *page* xi

1 An Introduction to Vortex Dynamics for Incompressible Fluid Flows 1
1.1 The Euler and the Navier–Stokes Equations 2
1.2 Symmetry Groups for the Euler and the Navier–Stokes Equations 3
1.3 Particle Trajectories 4
1.4 The Vorticity, a Deformation Matrix, and Some Elementary Exact Solutions 6
1.5 Simple Exact Solutions with Convection, Vortex Stretching, and Diffusion 13
1.6 Some Remarkable Properties of the Vorticity in Ideal Fluid Flows 20
1.7 Conserved Quantities in Ideal and Viscous Fluid Flows 24
1.8 Leray's Formulation of Incompressible Flows and Hodge's Decomposition of Vector Fields 30
1.9 Appendix 35
Notes for Chapter 1 41
References for Chapter 1 42

2 The Vorticity-Stream Formulation of the Euler and the Navier-Stokes Equations 43
2.1 The Vorticity-Stream Formulation for 2D Flows 44
2.2 A General Method for Constructing Exact Steady Solutions to the 2D Euler Equations 46
2.3 Some Special 3D Flows with Nontrivial Vortex Dynamics 54
2.4 The Vorticity-Stream Formulation for 3D Flows 70
2.5 Formulation of the Euler Equation as an Integrodifferential Equation for the Particle Trajectories 81
Notes for Chapter 2 84
References for Chapter 2 84

3 Energy Methods for the Euler and the Navier–Stokes Equations 86
3.1 Energy Methods: Elementary Concepts 87

3.2 Local-in-Time Existence of Solutions by Means of Energy Methods 96
3.3 Accumulation of Vorticity and the Existence of Smooth Solutions Globally in Time 114
3.4 Viscous-Splitting Algorithms for the Navier–Stokes Equation 119
3.5 Appendix for Chapter 3 129
Notes for Chapter 3 133
References for Chapter 3 134

4 The Particle-Trajectory Method for Existence and Uniqueness of Solutions to the Euler Equation 136
4.1 The Local-in-Time Existence of Inviscid Solutions 138
4.2 Link between Global-in-Time Existence of Smooth Solutions and the Accumulation of Vorticity through Stretching 146
4.3 Global Existence of 3D Axisymmetric Flows without Swirl 152
4.4 Higher Regularity 155
4.5 Appendixes for Chapter 4 158
Notes for Chapter 4 166
References for Chapter 4 167

5 The Search for Singular Solutions to the 3D Euler Equations 168
5.1 The Interplay between Mathematical Theory and Numerical Computations in the Search for Singular Solutions 170
5.2 A Simple 1D Model for the 3D Vorticity Equation 173
5.3 A 2D Model for Potential Singularity Formation in 3D Euler Equations 177
5.4 Potential Singularities in 3D Axisymmetric Flows with Swirl 185
5.5 Do the 3D Euler Solutions Become Singular in Finite Times? 187
Notes for Chapter 5 188
References for Chapter 5 188

6 Computational Vortex Methods 190
6.1 The Random-Vortex Method for Viscous Strained Shear Layers 192
6.2 2D Inviscid Vortex Methods 208
6.3 3D Inviscid-Vortex Methods 211
6.4 Convergence of Inviscid-Vortex Methods 216
6.5 Computational Performance of the 2D Inviscid-Vortex Method on a Simple Model Problem 227
6.6 The Random-Vortex Method in Two Dimensions 232
6.7 Appendix for Chapter 6 247
Notes for Chapter 6 253
References for Chapter 6 254

7 Simplified Asymptotic Equations for Slender Vortex Filaments 256
7.1 The Self-Induction Approximation, Hasimoto's Transform, and the Nonlinear Schrödinger Equation 257

7.2 Simplified Asymptotic Equations with Self-Stretch for a Single Vortex Filament 262
7.3 Interacting Parallel Vortex Filaments – Point Vortices in the Plane 278
7.4 Asymptotic Equations for the Interaction of Nearly Parallel Vortex Filaments 281
7.5 Mathematical and Applied Mathematical Problems Regarding Asymptotic Vortex Filaments 300
Notes for Chapter 7 301
References for Chapter 7 301

8 Weak Solutions to the 2D Euler Equations with Initial Vorticity in L^∞ 303
8.1 Elliptical Vorticies 304
8.2 Weak L^∞ Solutions to the Vorticity Equation 309
8.3 Vortex Patches 329
8.4 Appendix for Chapter 8 354
Notes for Chapter 8 356
References for Chapter 8 356

9 Introduction to Vortex Sheets, Weak Solutions, and Approximate-Solution Sequences for the Euler Equation 359
9.1 Weak Formulation of the Euler Equation in Primitive-Variable Form 361
9.2 Classical Vortex Sheets and the Birkhoff–Rott Equation 363
9.3 The Kelvin–Helmholtz Instability 367
9.4 Computing Vortex Sheets 370
9.5 The Development of Oscillations and Concentrations 375
Notes for Chapter 9 380
References for Chapter 9 380

10 Weak Solutions and Solution Sequences in Two Dimensions 383
10.1 Approximate-Solution Sequences for the Euler and the Navier–Stokes Equations 385
10.2 Convergence Results for 2D Sequences with L^1 and L^p Vorticity Control 396
Notes for Chapter 10 403
References for Chapter 10 403

11 The 2D Euler Equation: Concentrations and Weak Solutions with Vortex-Sheet Initial Data 405
11.1 Weak-* and Reduced Defect Measures 409
11.2 Examples with Concentration 411
11.3 The Vorticity Maximal Function: Decay Rates and Strong Convergence 421
11.4 Existence of Weak Solutions with Vortex-Sheet Initial Data of Distinguished Sign 432
Notes for Chapter 11 448
References for Chapter 11 448

12 Reduced Hausdorff Dimension, Oscillations, and Measure-Valued Solutions of the Euler Equations in Two and Three Dimensions 450
12.1 The Reduced Hausdorff Dimension 452
12.2 Oscillations for Approximate-Solution Sequences without L^1 Vorticity Control 472
12.3 Young Measures and Measure-Valued Solutions of the Euler Equations 479
12.4 Measure-Valued Solutions with Oscillations and Concentrations 492
Notes for Chapter 12 496
References for Chapter 12 496

13 The Vlasov–Poisson Equations as an Analogy to the Euler Equations for the Study of Weak Solutions 498
13.1 The Analogy between the 2D Euler Equations and the 1D Vlasov–Poisson Equations 502
13.2 The Single-Component 1D Vlasov–Poisson Equation 511
13.3 The Two-Component Vlasov–Poisson System 524
Note for Chapter 13 541
References for Chapter 13 541

Index 543

Preface

Vorticity is perhaps the most important facet of turbulent fluid flows. This book is intended to be a comprehensive introduction to the mathematical theory of vorticity and incompressible flow ranging from elementary introductory material to current research topics. Although the contents center on mathematical theory, many parts of the book showcase a modern applied mathematics interaction among rigorous mathematical theory, numerical, asymptotic, and qualitative simplified modeling, and physical phenomena. The interested reader can see many examples of this symbiotic interaction throughout the book, especially in Chaps. 4–9 and 13. The authors hope that this point of view will be interesting to mathematicians as well as other scientists and engineers with interest in the mathematical theory of incompressible flows.

The first seven chapters comprise material for an introductory graduate course on vorticity and incompressible flow. Chapters 1 and 2 contain elementary material on incompressible flow, emphasizing the role of vorticity and vortex dynamics together with a review of concepts from partial differential equations that are useful elsewhere in the book. These formulations of the equations of motion for incompressible flow are utilized in Chaps. 3 and 4 to study the existence of solutions, accumulation of vorticity, and convergence of numerical approximations through a variety of flexible mathematical techniques. Chapter 5 involves the interplay between mathematical theory and numerical or quantitative modeling in the search for singular solutions to the Euler equations. In Chap. 6, the authors discuss vortex methods as numerical procedures for incompressible flows; here some of the exact solutions from Chaps. 1 and 2 are utilized as simplified models to study numerical methods and their performance on unambiguous test problems. Chapter 7 is an introduction to the novel equations for interacting vortex filaments that emerge from careful asymptotic analysis.

The material in the second part of the book can be used for a graduate course on the theory for weak solutions for incompressible flow with an emphasis on modern applied mathematics. Chapter 8 is an introduction to the mildest weak solutions such as patches of vorticity in which there is a complete and elegant mathematical theory. In contrast, Chap. 9 involves a discussion of subtle theoretical and computational issues involved with vortex sheets as the most singular weak solutions in two-space dimensions with practical significance. This chapter also provides a pedagogical introduction to the mathematical material on weak solutions presented in Chaps. 10–12.

Chapter 13 involves a theoretical and computational study of the one-dimensional Vlasov–Poisson equations, which serves as a simplified model in which many of the unresolved issues for weak solutions of the Euler equations can be answered in an explicit and unambiguous fashion.

This book is a direct outgrowth of several extensive lecture courses by Majda on these topics at Princeton University during 1985, 1988, 1990, and 1993, and at the Courant Institute in 1995. This material has been supplemented by research expository contributions based on both the authors' work and on other current research.

Andrew Majda would like to thank many former students in these courses who contributed to the write-up of earlier versions of the notes, especially Dongho Chae, Richard Dziurzynski, Richard McLaughlin, David Stuart, and Enrique Thomann. In addition, many friends and scientific collaborators have made explicit or implicit contributions to the material in this book. They include Tom Beale, Alexandre Chorin, Peter Constantin, Rupert Klein, and George Majda. Ron DiPerna was a truly brilliant mathematician and wonderful collaborator who passed away far too early in his life; it is a privilege to give an exposition of aspects of our joint work in the later chapters of this book.

We would also like to thank the following people for their contributions to the development of the manuscript through proofreading and help with the figures and typesetting: Michael Brenner, Richard Clelland, Diego Cordoba, Weinan E, Pedro Embid, Andrew Ferrari, Judy Horowitz, Benjamin Jones, Phyllis Kronhaus, Monika Nitsche, Mary Pugh, Philip Riley, Thomas Witelski, and Yuxi Zheng. We thank Robert Krasny for providing us with Figures 9.4 and 9.5 in Chap. 9.