1

An Introduction to Vortex Dynamics for Incompressible Fluid Flows

In this book we study incompressible high Reynolds numbers and incompressible inviscid flows. An important aspect of such fluids is that of *vortex dynamics*, which in lay terms refers to the interaction of local swirls or eddies in the fluid. Mathematically we analyze this behavior by studying the rotation or *curl* of the velocity field, called the *vorticity*. In this chapter we introduce the Euler and the Navier–Stokes equations for incompressible fluids and present elementary properties of the equations. We also introduce some elementary examples that both illustrate the kind of phenomena observed in hydrodynamics and function as building blocks for more complicated solutions studied in later chapters of this book.

This chapter is organized as follows. In Section 1.1 we introduce the equations, relevant physical quantities, and notation. Section 1.2 presents basic symmetry groups of the Euler and the Navier–Stokes equations. In Section 1.3 we discuss the motion of a particle that is carried with the fluid. We show that the particle-trajectory map leads to a natural formulation of how quantities evolve with the fluid. Section 1.4 shows how locally an incompressible field can be approximately decomposed into translation, rotation, and deformation components. By means of exact solutions, we show how these simple motions interact in solutions to the Euler or the Navier–Stokes equations. Continuing in this fashion, Section 1.5 examines exact solutions with shear, vorticity, convection, and diffusion. We show that although deformation can increase vorticity, diffusion can balance this effect. Inviscid fluids have the remarkable property that vorticity is transported (and sometimes stretched) along streamlines. We discuss this in detail in Section 1.6, including the fact that vortex lines move with the fluid and circulation over a closed curve is conserved. This is an example of a quantity that is locally conserved. In Section 1.7 we present a number of global quantities, involving spatial integrals of functions of the solution, such as the kinetic energy, velocity, and vorticity flux, that are conserved for the Euler equation. In the case of Navier–Stokes equations, diffusion causes some of these quantities to dissipate. Finally, in Section 1.8, we show that the incompressibility condition leads to a natural reformulation of the equations (which are due to Leray) in which the pressure term can be replaced with a nonlocal bilinear function of the velocity field. This is the sense in which the pressure plays the role of a Lagrange multiplier in the evolution equation. The appendix of this chapter reviews the Fourier series and the Fourier transform

(Subsection 1.9.1), elementary properties of the Poisson equation (Subsection 1.9.2), and elementary properties of the heat equation (Subsection 1.9.3).

1.1. The Euler and the Navier–Stokes Equations

Incompressible flows of homogeneous fluids in all of space $\mathbb{R}^N$, $N = 2, 3$, are solutions of the system of equations

$$\frac{Dv}{Dt} = -\nabla p + \nu \Delta v, \tag{1.1}$$

$$\operatorname{div} v = 0, \qquad (x, t) \in \mathbb{R}^N \times [0, \infty), \tag{1.2}$$

$$v|_{t=0} = v_0, \qquad x \in \mathbb{R}^N, \tag{1.3}$$

where $v(x, t) \equiv (v^1, v^2, \ldots, v^N)^t$ is the fluid velocity, $p(x, t)$ is the scalar pressure, D/Dt is the convective derivative (i.e., the derivative along particle trajectories),

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{j=1}^{N} v^j \frac{\partial}{\partial x_j}, \tag{1.4}$$

and div is the divergence of a vector field,

$$\operatorname{div} v = \sum_{j=1}^{N} \frac{\partial v^j}{\partial x_j}. \tag{1.5}$$

The gradient operator ∇ is

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_N} \right)^t, \tag{1.6}$$

and the Laplace operator Δ is

$$\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}. \tag{1.7}$$

A given kinematic constant viscosity $\nu \geq 0$ can be viewed as the reciprocal of the Reynolds number R_e. For $\nu > 0$, Eq. (1.1) is called the *Navier–Stokes* equation; for $\nu = 0$ it reduces to the *Euler equation*. These equations follow from the conservation of momentum for a continuum (see, e.g., Chorin and Marsden, 1993). Equation (1.2) expresses the incompressibility of the fluid (see Proposition 1.4). The initial value problem [Eqs. (1.1)–(1.3)] is unusual because it contains the time derivatives of only three out of the four unknown functions. In Section 1.8 we show that the pressure $p(x, t)$ plays the role of a Lagrange multiplier and that a nonlocal operator in $\mathbb{R}^N$ determines the pressure from the velocity $v(x, t)$.

This book often considers examples of incompressible fluid flows in the *periodic* case, i.e.,

$$v(x + e_i, t) = v(x, t), \qquad i = 1, 2, \ldots, N, \tag{1.8}$$

for all x and $t \geq 0$, where e_i are the standard basis vectors in $\mathbb{R}^N$, $e_1 = (1, 0, \ldots,)^t$, etc. Periodic flows provide prototypical examples for fluid flows in bounded domains $\Omega \subset \mathbb{R}^N$. In this case the bounded domain Ω is the N-dimensional torus T^N. Flows on the torus serve as especially good elementary examples because we have Fourier series techniques (see Subsection 1.9.1) for computing explicit solutions. We make use of these methods, e.g., in Proposition 1.18 (the Hodge decomposition of T^N) in this chapter and repeatedly throughout this book.

In many applications, e.g., predicting hurricane paths or controlling large vortices shed by jumbo jets, the viscosity ν is very small: $\nu \sim 10^{-6} - 10^{-3}$. Thus we might anticipate that the behavior of inviscid solutions (with $\nu = 0$) would give a lot of insight into the behavior of viscous solutions for a small viscosity $\nu \ll 1$. In this chapter and Chap. 2 we show this to be true for explicit examples. In Chap. 3 we prove this result for general solutions to the Navier–Stokes equation in $\mathbb{R}^N$ (see Proposition (3.2).

1.2. Symmetry Groups for the Euler and the Navier–Stokes Equations

Here we list some elementary symmetry groups for solutions to the Euler and the Navier–Stokes equations. By straightforward inspection we get the following proposition.

Proposition 1.1. *Symmetry Groups of the Euler and the Navier–Stokes Equations. Let v, p be a solution to the Euler or the Navier–Stokes equations. Then the following transformations also yield solutions:*

(i) Galilean invariance: For any constant-velocity vector $\mathbf{c} \in \mathbb{R}^N$,

$$\begin{aligned} v_{\mathbf{c}}(x, t) &= v(x - \mathbf{c}t, t) + \mathbf{c}, \\ p_{\mathbf{c}}(x, t) &= p(x - \mathbf{c}t, t) \end{aligned} \tag{1.9}$$

is also a solution pair.

(ii) Rotation symmetry: for any rotation matrix $\mathcal{Q}$ ($\mathcal{Q}^t = \mathcal{Q}^{-1}$),

$$\begin{aligned} v_{\mathcal{Q}}(x, t) &= \mathcal{Q}^t v(\mathcal{Q}x, t), \\ p_{\mathcal{Q}}(x, t) &= p(\mathcal{Q}x, t) \end{aligned} \tag{1.10}$$

is also a solution pair.

(iii) Scale invariance: for any $\lambda, \tau \in \mathbb{R}$,

$$v_{\lambda,\tau}(x, t) = \frac{\lambda}{\tau} v\left(\frac{x}{\lambda}, \frac{t}{\tau}\right), \qquad p_{\lambda,\tau}(x, t) = \frac{\lambda^2}{\tau^2} p\left(\frac{x}{\lambda}, \frac{t}{\tau}\right), \tag{1.11}$$

is a solution pair to the Euler equation, and for any $\tau \in \mathbb{R}^+$,

$$v_\tau(x, t) = \tau^{-1/2} v\left(\frac{x}{\tau^{1/2}}, \frac{t}{\tau}\right), \qquad p_\tau(x, t) = \tau^{-1} p\left(\frac{x}{\tau^{1/2}}, \frac{t}{\tau}\right), \tag{1.12}$$

is a solution pair to the Navier–Stokes equation.

We note that scaling transformations determine the two-parameter symmetry group given in Eqs. (1.11) for the Euler equation. The introduction of viscosity $\nu > 0$, however, restricts this symmetry group to the one-parameter group given in Eqs. (1.12) for the Navier–Stokes equation.

1.3. Particle Trajectories

An important construction used throughout this book is the *particle-trajectory mapping* $X(\cdot, t)\colon \alpha \in \mathbb{R}^N \to X(\alpha, t) \in \mathbb{R}^N$. Given a fluid velocity $v(x, t)$, $X(\alpha, t) = (X_1, X_2, \ldots, X_N)^t$ is the location at time t of a fluid particle initially placed at the point $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)^t$ at time $t = 0$. The following nonlinear ordinary differential equation (ODE) defines particle-trajectory mapping:

$$\frac{dX}{dt}(\alpha, t) = v(X(\alpha, t), t), \qquad X(\alpha, 0) = \alpha. \tag{1.13}$$

The parameter α is called the Lagrangian particle marker. The particle-trajectory mapping X has a useful interpretation: An initial domain $\Omega \subset \mathbb{R}^N$ in a fluid evolves in time to $X(\Omega, t) = \{X(\alpha, t)\colon\ \alpha \in \Omega\}$, with the vector v tangent to the particle trajectory (see Fig. 1.1).

Next we review some elementary properties of $X(\cdot, t)$. We define the Jacobian of this transformation by

$$J(\alpha, t) = \det(\nabla_\alpha X(\alpha, t)). \tag{1.14}$$

We use subscripts to denote partial derivatives and variables of differential operators, e.g., $f_t = \partial/\partial t f$, $\nabla_\alpha = [(\partial/\partial\alpha_1), \ldots, (\partial/\partial\alpha_N)]$. The time evolution of the Jacobian J satisfies the following proposition.

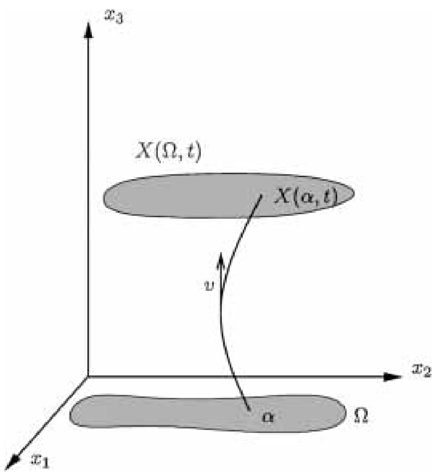

Figure 1.1. The particle-trajectory map.

Proposition 1.2. *Let $X(\cdot, t)$ be a particle-trajectory mapping of a smooth velocity field $v \in \mathbb{R}^N$. Then*

$$\frac{\partial J}{\partial t} = (\operatorname{div}_x v)|_{(X(\alpha,t),t)} J(\alpha, t). \tag{1.15}$$

We also frequently need a formula to determine the rate of change of a given function $f(x, t)$ in a domain $X(\Omega, t)$ moving with the fluid. This calculus formula, called the transport formula, is the following proposition.

Proposition 1.3. *(The Transport Formula). Let $\Omega \subset \mathbb{R}^N$ be an open, bounded domain with a smooth boundary, and let X be a given particle-trajectory mapping of a smooth velocity field v. Then for any smooth function $f(x, t)$,*

$$\frac{d}{dt} \int_{X(\Omega,t)} f \, dx = \int_{X(\Omega,t)} [f_t + \operatorname{div}_x(f v)] dx. \tag{1.16}$$

We give the proofs of Propositions 1.2 and 1.3 below. As an immediate application of these results, we note that either $J(\alpha, t) = 1$ or $\operatorname{div} v = 0$ implies incompressibility.

Definition 1.1. *A flow $X(\cdot, t)$ is incompressible if for all subdomains Ω with smooth boundaries and any $t > 0$ the flow is volume preserving:*

$$\operatorname{vol} X(\Omega, t) = \operatorname{vol} \Omega.$$

Applying the transport formula in Eq. (1.16) for $f \equiv 1$, we get $\operatorname{div} v = 0$. Moreover, then Eq. (1.15) yields $J(\alpha, t) = J(\alpha, 0) = 1$. We state this as a proposition below.

Proposition 1.4. *For smooth flows the following three conditions are equivalent:*

(i) *a flow is incompressible, i.e.,* $\forall \Omega \subset \mathbb{R}^N$, $t \geq 0$ $\operatorname{vol} X(\Omega, t) = \operatorname{vol} \Omega$,
(ii) $\operatorname{div} v = 0$,
(iii) $J(\alpha, t) = 1$.

Now we give the proof of Proposition 1.2.

Proof of Proposition 1.2. Because the determinant is multilinear in columns (rows), we compute the time derivative

$$\frac{\partial J}{\partial t} = \frac{\partial}{\partial t} \det \left[\frac{\partial X^i}{\partial \alpha_j}(\alpha, t) \right] = \sum_{i,j} A_i^j \frac{\partial}{\partial t} \frac{\partial X^i}{\partial \alpha_j}(\alpha, t),$$

where A_i^j is the minor of the element $\partial X^i / \partial \alpha_j$ of the matrix $\nabla_\alpha X$. The minors satisfy the well-known identity

$$\sum_j \frac{\partial X^k}{\partial \alpha_j} A_i^j = \delta_i^k J, \qquad \text{where} \quad \delta_i^k = \begin{cases} 1, & k = i \\ 0, & k \neq i \end{cases}.$$

The definition of the particle trajectories in Eq. (1.13) then gives

$$\frac{\partial J}{\partial t} = \sum_{i,j,k} A_i^j \frac{\partial X^k}{\partial \alpha_j} v^i_{x_k} = \sum_{i,k} v^i_{x_k} \delta_i^k J = J \operatorname{div} v. \qquad \square$$

Finally we give the proof of Proposition 1.3.

Proof of Proposition 1.3. By the change of variables $\alpha \to X(\alpha, t)$, we reduce the integration over the moving domain $X(\Omega, t)$ to the integration over the fixed domain Ω:

$$\int_{X(\Omega,t)} f(x,t)dx = \int_\Omega f(X(\alpha,t),t)J(\alpha,t)d\alpha.$$

The definition in Eq. (1.13) implies that

$$\begin{aligned} \frac{d}{dt}\int_{X(\Omega,t)} f\,dx &= \int_\Omega \left[\left(\frac{\partial f}{\partial t} + \frac{dX}{dt}\cdot\nabla f\right)J + f\frac{\partial J}{\partial t}\right]d\alpha \\ &= \int_\Omega \left(\frac{\partial f}{\partial t} + v_x\cdot\nabla f + f\operatorname{div}_x v\right)J\,d\alpha \\ &= \int_{X(\Omega,t)} \left[\frac{\partial f}{\partial t} + \operatorname{div}_x(fv)\right]dx. \qquad \square \end{aligned}$$

1.4. The Vorticity, a Deformation Matrix, and Some Elementary Exact Solutions

First we determine a simple local description for an incompressible fluid flow. Every smooth velocity field $v(x,t)$ has a Taylor series expansion at a fixed point (x_0, t_0):

$$v(x_0 + h, t_0) = v(x_0, t_0) + (\nabla v)(x_0, t_0)h + \mathcal{O}(h^2), \qquad h \in \mathbb{R}^3. \tag{1.17}$$

The 3×3 matrix $\nabla v = (v^i_{x_j})$ has a *symmetric* part $\mathcal{D}$ and an *antisymmetric* part Ω:

$$\mathcal{D} = \frac{1}{2}(\nabla v + \nabla v^t), \tag{1.18}$$

$$\Omega = \frac{1}{2}(\nabla v - \nabla v^t). \tag{1.19}$$

$\mathcal{D}$ is called the deformation or rate-of-strain matrix, and Ω is called the rotation matrix. If the flow is incompressible, div $v = 0$, then the trace tr $\mathcal{D} = \sum_i d_{ii} = 0$. Moreover, the vorticity ω of the vector field v,

$$\omega = \operatorname{curl} v \equiv \left(\frac{\partial v^3}{\partial x_2} - \frac{\partial v^2}{\partial x_3}, \frac{\partial v^1}{\partial x_3} - \frac{\partial v^3}{\partial x_1}, \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}\right)^t, \tag{1.20}$$

satisfies

$$\Omega h = \frac{1}{2}\omega \times h, \qquad \forall h \in \mathbb{R}^3. \tag{1.21}$$

Using the Taylor series expression (1.17) and the new definitions, we have Lemma 1.1

Lemma 1.1. *To linear order in* $(|x - x_0|)$, *every smooth incompressible velocity field* $v(x, t)$ *is the (unique) sum of three terms:*

$$v(x, t_0) = v(x_0, t_0) + \frac{1}{2}\omega \times (x - x_0) + \mathcal{D}(x - x_0), \tag{1.22}$$

where $\mathcal{D}$ *is the (symmetric) deformation matrix with* $\operatorname{tr} \mathcal{D} = 0$ *and* ω *is the vorticity.*

The successive terms in Eq. (1.22) have a natural physical interpretation in terms of translation, rotation, and deformation. Retaining only the term $v(x_0, t_0)$ in Eq. (1.22) gives

$$X(\alpha, t) = \alpha + v(x_0, t_0)(t - t_0),$$

which describes an *infinitesimal translation*.

By a rotation of axes, without loss of generality, we can assume that $\omega = (0, 0, \omega)^t$, so

$$\omega \times (x - x_0) = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (x - x_0).$$

Thus retaining only this term in the velocity for the particle-trajectory equation gives the particle trajectories $X = (X', X_3)$ as

$$X'(\alpha, t) = x_0' + \mathcal{Q}\left(\frac{1}{2}\omega t\right)(x' - x_0'), \qquad X_3(\alpha, t) = x_0^3,$$

where $\mathcal{Q}$ is the rotation matrix in the $x_1 - x_2$ plane:

$$\mathcal{Q}(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

These trajectories are circles on the $x_1 - x_2$ plane, so the second term $\frac{1}{2}\omega \times (x - x_0)$ in Eq. (1.22) is an infinitesimal rotation in the direction of ω with angular velocity $\frac{1}{2}|\omega|$.

Finally, because $\mathcal{D}$ is a symmetric matrix, there is a rotation matrix $\mathcal{Q}$ so that $\mathcal{Q}\mathcal{D}\mathcal{Q}^t = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3)$. Moreover, traces are invariant under similarity transformations so that $\gamma_1 + \gamma_2 + \gamma_3 = 0$. Thus, without loss of generality, assume that

$$\mathcal{D} = \operatorname{diag}[\gamma_1, \gamma_2, -(\gamma_1 + \gamma_2)].$$

Retaining only the term $\mathcal{D}(x - x_0)$ from Eq. (1.22) in the particle-trajectory equation yields

$$X(\alpha, t) = x_0 + \begin{bmatrix} e^{\gamma_1(t - t_0)} & 0 & 0 \\ 0 & e^{\gamma_2(t - t_0)} & 0 \\ 0 & 0 & e^{-(\gamma_1 + \gamma_2)(t - t_0)} \end{bmatrix} (\alpha - x_0).$$

For example, if we set $\gamma_1, \gamma_2 > 0$, $x_0 = 0$, the fluid is compressed along the x_1–x_2 plane but stretched along the x_3 axis, creating a jet. This corresponds to a sharp deformation of the fluid. Thus the third term in Eq. (1.22) represents an infinitesimal deformation velocity in the direction $(x - x_0)$.

We have just proved the following corollary.

Corollary 1.1. *To linear order in* $(|x - x_0|)$, *every incompressible velocity field* $v(x, t)$ *is the sum of infinitesimal translation, rotation, and deformation velocities.*

A large part of this book addresses the interactions among these three contributions to the velocity field. To illustrate the interaction between a vorticity and a deformation, we now derive a large class of exact solutions for both the Euler and the Navier–Stokes equations.

Proposition 1.5. *Let* $\mathcal{D}(t)$ *be a real, symmetric,* 3×3 *matrix with* $\operatorname{tr} \mathcal{D}(t) = 0$. *Determine the vorticity* $\omega(t)$ *from the ODE on* $\mathbb{R}^3$,

$$\frac{d\omega}{dt} = \mathcal{D}(t)\omega, \qquad \omega|_{t=0} = \omega_0 \in \mathbb{R}^3, \tag{1.23}$$

and the antisymmetric matrix Ω *by means of the formula* $\Omega h = \frac{1}{2}\omega \times h$. *Then*

$$\begin{aligned} v(x, t) &= \frac{1}{2}\omega(t) \times x + \mathcal{D}(t)x, \\ p(x, t) &= -\frac{1}{2}[\mathcal{D}_t(t) + \mathcal{D}^2(t) + \Omega^2(t)]x \cdot x \end{aligned} \tag{1.24}$$

are exact solutions to the three-dimensional (3D) Euler and the Navier–Stokes equations.

The solutions in Eqs. (1.24) can be trivially generalized by use of the Galilean invariance (see Proposition 1.1). Because the pressure p has a quadratic behavior in x, these solutions have a direct physical meaning only locally in space and time. Also, because the velocity is linear in x, the effects of viscosity do not alter these solutions. Nevertheless, these solutions model the typical local behavior of incompressible flows.

Before proving this proposition, first we give some examples of the exact solutions in Eqs. (1.24) that illustrate the interactions between a rotation and a deformation.

Example 1.1. Jet Flows. Taking $\omega_0 = 0$ and $\mathcal{D} = \operatorname{diag}(-\gamma_1, -\gamma_2, \gamma_1 + \gamma_2)$, $\gamma_j > 0$, from Eqs. (1.23 and 1.24) we get

$$v(x, t) = [-\gamma_1 x_1, -\gamma_2 x_2, (\gamma_1 + \gamma_2)x_3]^t. \tag{1.25}$$

This flow is irrotational, $\omega = 0$, and forms two jets along the positive and the negative

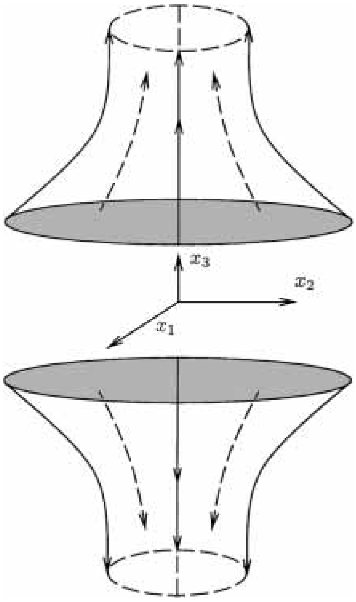

Figure 1.2. A jet flow as described in Example 1.1.

directions of the x_3 axis, with particle trajectories (see Fig. 1.2)

$$X(\alpha, t) = \begin{bmatrix} e^{-\gamma_1 t} & 0 & 0 \\ 0 & e^{-\gamma_2 t} & 0 \\ 0 & 0 & e^{(\gamma_1+\gamma_2)t} \end{bmatrix} \alpha.$$

Observe that

$$\left(X_1^2 + X_2^2\right)(\alpha, t) = e^{-2(\gamma_1+\gamma_2)t}\left(\alpha_1^2 + \alpha_2^2\right),$$

so the distance of a given fluid particle to the x_3 axis decreases exponentially in time. A jet flow is one type of axisymmetric flow without swirl, which will be discussed in Subsection 2.3.3 of Chap. 2.

Example 1.2. Strain Flows. Now taking $\omega_0 = 0$ and $\mathcal{D} = \text{diag}(-\gamma, \gamma, 0)$, $\gamma > 0$, from Eq. (1.25) we get

$$v(x, t) = (-\gamma x_1, \gamma x_2, 0)^t. \tag{1.26}$$

This flow is irrotational $\omega = 0$ and forms a strain flow (independent of x_3) with the particle trajectories $X = (X', X_3)$ (see Fig. 1.3):

$$X'(\alpha, t) = \begin{bmatrix} e^{-\gamma t} & 0 \\ 0 & e^{\gamma t} \end{bmatrix} \alpha', \qquad X_3(\alpha, t) = \alpha_3.$$

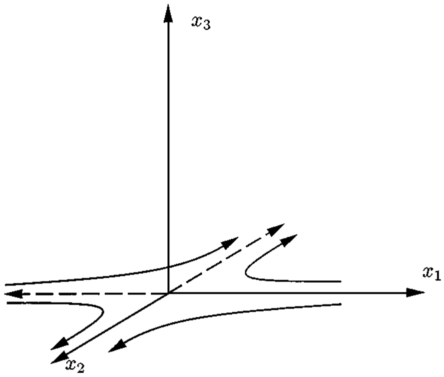

Figure 1.3. A strain flow as described in Example 1.2. This flow is independent of the variable x_3.

Example 1.3. A Vortex. Taking $\mathcal{D} = 0$ and $\omega_0 = (0, 0, \omega)^t$, from Eqs. (1.23) and (1.24) we get

$$v(x, t) = \left(-\frac{1}{2}\omega_0 x_2, \frac{1}{2}\omega_0 x_1, 0\right)^t. \tag{1.27}$$

This flow is a rigid rotation motion in the $x_1 - x_2$ plane, with the angular velocity $\frac{1}{2}\omega_0$ and the particle trajectories $X = (X', X_3)$, $X' = (X^1, X^2)$ (see Fig. 1.4):

$$X'(\alpha, t) = \mathcal{Q}\left(\frac{1}{2}\omega_0 t\right)\alpha', \qquad X_3(\alpha, t) = \alpha_3,$$

where $\mathcal{Q}$ is the 2×2 rotation matrix

$$\mathcal{Q}(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

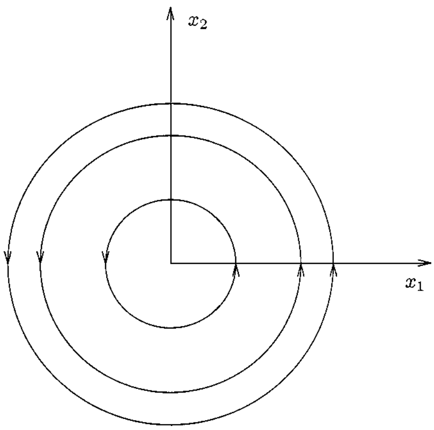

Figure 1.4. A two-dimensional vortex as described in Example 1.3.