1
Background

I shall not be satisfied unless I produce something that shall for a few days supersede the last fashionable novel on the tables of young ladies.

LORD THOMAS B. MACAULAY
on his *History of England*.

The term "stratified flow" is commonly used to denote the flow of "stratified fluid", or more correctly "density-stratified fluid", and it is so used here. In such fluids the density (mass per unit volume) varies with position in the fluid, and this variation is dynamically important. Normally this density variation is stable with lines of constant density oriented nearly horizontally, with lighter fluid above and heavier fluid below. The density variation may be continuous, as occurs in most of the atmosphere and ocean, or be concentrated in discontinuous interfaces, such as at the surface of the ocean. In many situations the variation in density is very small, but such variations may have a dominant effect on the flow if the small buoyancy forces are given sufficient time to act. This book is about the motion of such fluids caused by their flow over topographic features. There has been substantial progress in recent years in this area. It is now possible to view the subject as a whole, and to understand the relation between, for example, the flow of a river over a ridge or weir, and the flow of the atmosphere over a mountain range. Whilst some details remain to be resolved, a corner has been turned, and the subject may now be viewed from a new and broader perspective.

The terms "topography", used in the title, and the equally common "orography" are not equivalent. The first may be taken to mean any departure from a level surface, such as lumps and bumps in a laboratory experiment for example. "Orography", on the other hand, implies obstacles on the large scale such as mountain ranges, which affect stratified processes in the atmosphere, and excludes rocks, buildings and the like, which don't. Topography therefore includes orography, and we will use the first term here for generality and consistency, although the second may be more appropriate when relating to atmospheric flows.

The subject of stratified flows is relevant to meteorology, oceanography and environmental engineering in the broad sense, with specific applications ranging from flow around hills to flow under ice keels. However, the impetus for the recent progress has been driven by atmospheric considerations more than others. This is for two reasons. Firstly, it has become clear that a lack of understanding of the effects of mountain ranges and smaller topographic features on the atmosphere has been a significant impediment to the improvement in weather forecasting. The form drag (as distinct from surface frictional drag) of the topography is known to constitute about 50% of the total drag on the atmosphere (e.g. Palmer *et al*. 1986), and this drag is manifested in stratified effects such as internal gravity waves. The correction of this deficiency requires adequate parametrisation of sub-grid-scale topographic effects in large-scale models for numerical weather prediction. The conspicuous deficiencies have stimulated major atmospheric research programmes such as ALPEX and PYREX, as well as many smaller studies. Secondly, and on a smaller scale, increasing concerns about environmental issues such as atmospheric pollution and air quality have instigated a number of mesoscale field and other studies which aim to describe the motion of air in or around mountains, hills and valleys, collectively known as "complex terrain", in considerable detail (see Blumen 1990). However, field programmes do not usually provide mechanistic answers by themselves, because of the general sparseness, paucity and ambiguity of most field data sets. Consequently, most of the improved understanding in dynamics has instead come from analytical, numerical and laboratory studies, with the field data supplying confirmation of applicability to the atmosphere.

The effects of the Earth's rotation (i.e. the Coriolis force) have been excluded from the whole of this monograph, and this places some restrictions on the applicability of the material described to atmospheric flows. Specifically, it strictly applies to atmospheric flows that last for a few hours or less (significantly less than a pendulum day), and have length-scales of a few tens of kilometres or less. In the ocean, where fluid velocities are typically smaller by a factor of 100, the maximum relevant length-scale is smaller by the same factor and is typically several hundred metres. However, when the flow is constrained to a channel, as in narrow straits, estuaries and rivers, the effect of rotation on flow in the downstream direction can be small, even over quite large distances.

The remainder of this chapter is concerned with the basic equations and their applicability to the atmosphere, their relevant boundary conditions, some conservation equations that will be needed later, and some comments on terminology. Chapter 2 is concerned with the properties of topographically forced flows of a homogeneous fluid layer with a free surface, which is applicable to flow in a river or channel. This system constitutes, in a sense, the simplest example of a stratified fluid, but there is still plenty of scope for non-linear complexity here, and it provides a useful prototype for the more general stratification considered in subsequent chapters. In Chapter 3 the additional effects present when the layer is surmounted by a second layer are discussed. These effects on the lower layer may only amount to "reduced gravity" due to the density of the overlying layer if the latter is deep, the motion is inviscid, and it has long horizontal length-scales. On the other hand, if the upper layer is shallow and in motion, the character of the flow may be quite different from that of a single layer. An example of this is the important case of "exchange flows", where the two layers are flowing in opposite directions. In Chapter 4 we proceed from two layers to "many", and discuss the behaviour of disturbances in continuously stratified fluids *per se*, without considering the effect of topography specifically. For the most part, this concerns the general properties of small-amplitude internal gravity waves in stratified flows with shear. Chapters 5 and 6 are devoted to the effects of continuously stratified flow over two- and three-dimensional topography, respectively, and constitute the heart of the book, whereas Chapter 7 is concerned with the application of the material of the preceding chapters to laboratory modelling of flow over complex terrain, and the parametrisation of sub-grid-scale orographic effects in numerical models of the atmosphere.

1.1 Equations for fluid motion

The equations governing the motion of a stratified fluid are (see, for example, Batchelor 1967)

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -g\hat{\mathbf{z}} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}, \tag{1.1.1}$$

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \nabla\cdot\mathbf{u} = 0, \tag{1.1.2}$$

where $\mathbf{u} = (u, v, w)$ is the fluid velocity, ρ the fluid density, p the pressure, g the acceleration due to gravity, and ν the kinematic viscosity (assumed spatially uniform). $\mathrm{D}/\mathrm{D}t$ denotes the Lagrangian derivative with respect to time t, and $\hat{\mathbf{z}}$ is the unit vector vertically upwards. In most situations the viscous term is small, and may be neglected for present purposes. If the density of each very small particle of fluid always remains constant as it moves, in spite of variations in pressure, the fluid is *incompressible* and we have

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \tag{1.1.3}$$

which in conjunction with (1.1.2) implies

$$\nabla \cdot \mathbf{u} = 0. \tag{1.1.4}$$

An important quantity that characterises continuously stratified fluids is the *buoyancy frequency* N (formerly known as the Brunt-Väisälä frequency), which for incompressible fluids at rest is defined by

$$N^2 = -\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}. \tag{1.1.5}$$

N is the frequency of local unforced vertical oscillations of small amplitude, and is the highest frequency that local buoyancy-driven fluctuations may have. It therefore gives a characteristic time-scale for these motions (see, for example, Gill 1982 for more details). N has maximum values of about 10^{-2} rad/s in both the atmosphere and ocean, and this sets the minimum time-scale for the motions there that we will be considering. Under these circumstances, it is often appropriate to consider these motions as variations about a basic state, which is in hydrostatic equilibrium. The pressure and density fields may be expressed as

$$p = p_0(z) + p'(x, y, z, t), \quad \rho = \rho_0 + \rho'(x, y, z, t), \tag{1.1.6}$$

where p_0 and ρ_0 represent the values in hydrostatic equilibrium, and are related by

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g. \tag{1.1.7}$$

The inviscid equation of motion for the perturbations may then be written

$$(\rho_0 + \rho')\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -g\rho'\hat{\mathbf{z}} - \nabla p'. \tag{1.1.8}$$

For most purposes water is effectively incompressible, but in general air is not. The equilibrium state of the atmosphere, specified by $p_0(z)$, $\rho_0(z)$, is obtained from (1.1.7) and the equation of state for a uniform ideal gas, $p = \rho \mathcal{R} T$, for a specific temperature profile $T(z)$. Here $\mathcal{R} = \mathcal{R}^*/M$, where $\mathcal{R}^* = 8314.3$ joule/(kelvin kilomole) is the *Universal Gas Constant*, and M is the molecular weight of the gas in question. Since air is a mixture of gases, we require its mean or effective molecular weight, which for dry air is 28.97; and this gives $\mathcal{R} = 287$ joule/(kelvin kilogram). (For a more detailed discussion of the thermodynamics, including the effects of water vapour, the reader is referred to Wallace & Hobbs 1977.) If $T(z)$ (in kelvin) is taken to be uniform with height, as a simple approximation to the mean atmospheric profile, then $p_0(z) = \rho_0(z)\mathcal{R}T$, and (1.1.7) gives

$$p_0(z) = p_0(0)\,\mathrm{e}^{-z/H_s}, \quad \rho_0(z) = \rho_0(0)\,\mathrm{e}^{-z/H_s}, \tag{1.1.9}$$

where H_s is the *scale height*, defined by $H_s = \mathcal{R}T/g$. If T is 280 K, then $H_s = 8.2$ km. The equations (inviscid for simplicity) for the adiabatic motion of an ideal gas are (1.1.1) omitting viscous terms, (1.1.2) and

$$\frac{\mathrm{D}}{\mathrm{D}t}(p/\rho^\gamma) = 0, \quad \text{or} \quad \frac{\mathrm{D}\theta}{\mathrm{D}t} = 0, \tag{1.1.10}$$

where γ is the ratio of specific heats ($\gamma = 1.4$ for air) and θ is the *potential temperature* defined by $\theta = T(p_r/p)^{1-1/\gamma}$, where T is the actual temperature at pressure p, and p_r is the pressure at the reference level, usually 1000 mb. θ is the temperature an air parcel would have if transported adiabatically to the level where $p = p_r$. For a compressible ideal gas the buoyancy frequency N is given by

$$N^2 = \frac{g}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}z}. \tag{1.1.11}$$

There are circumstances under which air in motion satisfying these equations may be regarded as incompressible. These have been discussed in general terms by Batchelor (1967, pp. 168–9), and we consider them here in the special context of flow forced by topography. From (1.1.2, 1.1.10) we may obtain

$$\frac{1}{\gamma p}\frac{\mathrm{D}p}{\mathrm{D}t} = -\nabla \cdot \mathbf{u}, \tag{1.1.12}$$

and for the fluid to be effectively incompressible we require

$$\left|\frac{1}{\gamma p}\frac{\mathrm{D}p}{\mathrm{D}t}\right| \ll \left|\frac{\partial \mathbf{u}}{\partial x}\right|, \tag{1.1.13}$$

where the x-direction is chosen to be the principal direction of fluid flow (if there is one), and $\partial \mathbf{u}/\partial x$ is taken to be representative of the constituent terms in $\nabla \cdot \mathbf{u}$. If the variations in $\mathbf{u}$ due to the topography have magnitude U and horizontal length-scale L, and the variables are expressed relative to a hydrostatic basic state as in (1.1.8), then (1.1.13) becomes

$$\left| \frac{1}{\rho c_s^2} \frac{\partial p}{\partial t} - \frac{1}{2c_s^2} \frac{\mathrm{D}u^2}{\mathrm{D}t} - \frac{g' w}{c_s^2} \right| \ll \frac{U}{L}, \tag{1.1.14}$$

where $c_s = (\gamma p/\rho)^{1/2}$ is the speed of sound in an ideal gas, and $g' = \rho' g/\rho$. Now if the pressure varies with a frequency ω, the magnitude of these pressure variations will be of order $\rho L U \omega$, so that the first term on the left-hand side will be of order $LU\omega^2/c_s^2$, and the second of order $\omega U^2/c_s^2$. If the maximum local frequency of oscillations forced by flow over topography is the buoyancy frequency N, there are two possible choices for ω, namely N and the advective frequency, U_T/L, where U_T is the total fluid speed relative to the ground (and by implication, $U < U_T$). Admitting both of these magnitudes, it follows that the three terms on the left-hand side of (1.1.14) will all be individually smaller than the right-hand side if the following (not respective) conditions are met:

$$\frac{U_T^2}{c_s^2} \ll 1, \quad \frac{N^2 L^2}{c_s^2} \ll 1, \quad \frac{g' w}{c_s^2} \ll \frac{U}{L}. \tag{1.1.15}$$

For internal gravity waves forced by topography with frequency comparable with N, the appropriate length-scale is $L \sim U_T/N$; on the other hand, if L is a topographic length-scale that is much longer than U_T/N, the relevant frequency is not N but U_T/L. Either way, the second criterion reduces to the first, and since $c_s \sim 330$ m/s, these two conditions will be met provided $U_T < 100$ m/s. This is normally the case in the atmosphere, with the exception of jet streams in the upper troposphere.

For the last of (1.1.15), we may assume that $U/L \sim W/H$, where W is a typical magnitude for the vertical velocity and H is a typical vertical length-scale for its variation. We also have $g' \sim N^2 H$, so that the third requirement becomes

$$\frac{N^2 H^2}{c_s^2} \ll 1. \tag{1.1.16}$$

This criterion is satisfied in the atmosphere if $H < 10$ km, which is normally the case.

Approximation (1.1.16) may be avoided if $\partial \rho/\partial t$ is small, so that

(1.1.2) may be approximated by

$$\nabla \cdot (\rho \mathbf{u}) = 0. \tag{1.1.17}$$

This equation then takes the place of (1.1.4), and this approximation is termed the *anelastic approximation* (Ogura & Phillips 1962), because the system no longer supports sound waves. For its validity we require the first two of (1.1.15), namely $U_T^2/c_s^2 \ll 1$, as discussed above, and the assumption that the motion is buoyancy driven so that no frequency is greater than N. This approximation has been used in a number of numerical studies that will be described later. The solutions of this system are very similar in character to those of the incompressible system.

In addition to anelasticity and incompressibility, two further approximations will often be made, namely the Boussinesq approximation and the hydrostatic approximation. Each of these approximations provides a useful simplification of the dynamical equations in appropriate circumstances, without significantly affecting the character of the motion being studied. The conditions for their validity are as follows. We consider motion in a layer of fluid of total depth D and express ρ_0 as $\rho_0 = \bar{\rho} + \Delta\rho_0(z)$, where $\bar{\rho}$ is a mean density in the fluid layer. For imcompressible and anelastic motions, ρ' will be of the same order of magnitude as $\Delta\rho_0$. If we have $\Delta\rho_0/\bar{\rho} \ll 1$ in this layer, we may replace $\rho_0 + \rho'$ in (1.1.8) by its mean value $\bar{\rho}$, and incur an error of relative magnitude $\Delta\rho_0/\bar{\rho}$. The density variations are thereby neglected in the inertia term, but retained in the buoyancy-force term where they are multiplied by g. This constitutes the *Boussinesq approximation*. It is normally a good approximation for all watery fluids in geophysical situations, and the analytic simplification that it provides is valuable. For this approximation to be valid in the atmosphere we also require $D \ll H_s$, the scale height, which restricts us to the lowest 1 or 2 km of the atmosphere. This is not as restrictive as it sounds, however. Bretherton (1966) has shown that it is possible to transform the anelastic equations to a form in which the Boussinesq approximation remains valid for *linear* motions over a much deeper layer, provided that the vertical scale H of such motions is much less than H_s. Throughout this book, we will mostly be concerned with the Boussinesq form of the equations, since these apply to the lower part of the atmosphere and contain the essence of the topics being covered, without surplus complexity. For readers who are interested in upper atmosphere phenomena, most of these results are at least qualitatively valid in the non-Boussinesq system.

The *hydrostatic approximation* is valid when

$$\frac{\partial p'}{\partial z} \cong -\rho' g, \tag{1.1.18}$$

implying that the dynamical variations about the mean state are in hydrostatic balance, in addition to the mean flow state itself (from 1.1.7). For this to hold, we require

$$\left|\frac{\partial w}{\partial t} + \mathbf{u}\cdot\nabla w\right| \ll \left|\frac{\partial p'}{\partial z}\right|. \tag{1.1.19}$$

If U and W are taken to be the horizontal and vertical velocity variations on the length-scales L and H respectively, and with $W \sim UN/L$ and $H \sim U_{\mathrm{T}}/N$, it may be readily shown that (1.1.19) is satisfied if

$$\frac{U_{\mathrm{T}}}{NL} \ll 1. \tag{1.1.20}$$

For small-amplitude motions L is the horizontal scale of the forcing topography, but for large-amplitude motions we may have $L \sim U_{\mathrm{T}}/N$ also, so that hydrostaticity depends on amplitude, or steepness of the streamlines.

Leaving aside the effect of pressure, the principal factor causing the variation in the density of air is temperature, and the density of fluid particles may alter because of the molecular diffusion of heat from one to another. The density of sea water is affected by both temperature and salinity, which diffuse at different rates. For present purposes we will assume that the density of the fluid concerned is affected by the diffusion of a single component, notionally the temperature T, which satisfies

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \kappa_T \nabla^2 T, \tag{1.1.21}$$

where κ_T is the thermal diffusivity. For an incompressible fluid with an equation of state of the form $\rho = \rho(T)$, we may write

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\mathrm{d}\rho}{\mathrm{d}T}\frac{DT}{Dt} = \kappa\nabla^2\rho, \tag{1.1.22}$$

where κ is the diffusivity of density. If $\rho(T)$ is effectively linear over the range of interest with the form

$$\rho(T) = \rho_0[1 - \alpha(T - T_0)], \tag{1.1.23}$$

where ρ_0, α and T_0 are constant, then $\kappa = \kappa_T$. If molecular diffusion is important, therefore, (1.1.22) replaces (1.1.3) in the equations for the motion of an incompressible fluid.

In many cases it is convenient to approximate a continuously stratified fluid with a fluid that is made up of a superposition of distinct layers, each of uniform density. Normally, each of these homogeneous layers is thin relative to the length-scale of the motion, and the horizontal velocity within the layer may be supposed to be uniform through the local depth of the layer. Layered models are often used in conjunction with the hydrostatic approximation. Many phenomena of general interest and applicability may be described with a system consisting of only one or two layers, and Chapters 2 and 3 are concerned with such systems.

1.2 Boundary conditions

Most flows considered will be established by time-dependent development from some known initial state, so that one can see how a given state may be established. In many situations with stratified fluids, it is possible to obtain steady-state flow solutions by assuming known flow conditions upstream (or downstream). However, in two-dimensional (or nearly two-dimensional) situations, there are restrictions on the properties of the steady upstream velocity and density profiles that may exist for an obstacle of given height (see Chapter 5), and for those profiles that are permissible, it may not be obvious how such a flow could be established. It is not difficult to obtain steady solutions that appear to be unrealistic or unphysical. For this reason, we concentrate on initial-value problems where the motion is commenced from simple initial states.

We will mostly be considering isolated topography, that is, topography where the lower surface becomes horizontal at large distances from the origin (though not necessarily at the same level). Boundary conditions on the flow at large distances from the topography will embody the assumption that there is no inward-propagating energy from infinity, other than that specified (which in the cases considered here is nil).

Since we are mainly concerned with inviscid equations, we will mostly omit viscous stresses and diffusive effects here in the boundary conditions. In general, the lower boundary will be a rigid surface specified by $z = h(x, y)$, where $h \to 0$ as $x^2 + y^2 \to \infty$. For inviscid flow, the boundary condition is that the velocity component normal to this surface must vanish. For viscous flow, the tangential component must also vanish. For an isolated obstacle, the maximum value of

$h(x, y)$ is denoted by h_m, but this will be abbreviated to h in dimensionless ratios such as Nh/U.

The upper boundary condition may take one of two main forms. The first is the "finite-depth" form, where the stratified fluid is bounded above by either a rigid horizontal surface – a "rigid lid", or alternatively by an infinitely deep homogeneous layer. For the rigid lid, the boundary condition is

$$w = 0, \quad \text{at } z = D. \tag{1.2.1}$$

There is no existing term for the upper boundary of a fluid surmounted by a deep homogeneous layer, and it is defined here to be a *pliant surface* or *pliant boundary*. The appropriate boundary conditions for the stratified fluid with a pliant surface (a material surface of fluid particles) at $z = d = D + \eta(x, y, t)$, with homogeneous fluid above it, are

$$\left. \begin{array}{l} w = \dfrac{\mathrm{D}\eta}{\mathrm{D}t}, \quad (1.2.2) \\ p \text{ continuous} \quad (1.2.3) \end{array} \right\} \text{at } z = D + \eta.$$

If the fluid above is immiscible with the fluid below, surface tension forces are present, but these are ignored in this work. If the density of the upper homogeneous fluid is zero the pliant surface is termed a *free surface*, and (1.2.3) becomes $p = 0$.

The second form of upper boundary condition, the "infinite depth" form, applies to an infinitely deep stratified fluid where internal wave energy may propagate to great heights. This condition is a "radiation condition", which specifies that the waves radiate "out the top" without reflection. This means that there is no downward propagation of wave energy above a certain level ($z = D$, say).

1.3 Conservation relations

For the purposes of this section we take more general forms of the equations for incompressible flow by adding external forcing and heat gain (or loss) terms, so that (1.1.1) and (1.1.3) become

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -g\hat{\mathbf{z}} - \frac{1}{\rho}\nabla p + \mathbf{F}, \tag{1.3.1}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\dot{H}, \tag{1.3.2}$$

and (1.1.2) still applies. $\mathbf{F}$ represents some general and unspecified