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HAUSDORFF MEASURES

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CONTENTS

Cambridge University Press
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C. A. Rogers
Frontmatter
More information

FOREWORD

K.J. Falconer

In this foreword, citations by date refer to the end of the main book, and citations by initials in square brackets [ ] refer to (generally more recent) books and papers listed at the end of the foreword.

§1 Introduction

I am delighted that Professor Rogers' book *Hausdorff Measures* is to be republished and it is a great pleasure to write a foreword to the new edition. The book has had a profound influence, both directly and indirectly, on geometric measure theory and on the very many areas of mathematics that relate to Hausdorff measures and dimensions.

Professor Rogers' two books, this one and *Packing and Covering* [Rog1], have both become 'classics' and remain indispensable to research workers in the subjects. They are self-contained and lead the reader from first degree level to the frontiers of research. The books are notable for their clarity and precision and the detailed proofs make them especially suitable for the student.

This foreword attempts several things. It indicates the historical and mathematical context of Hausdorff measures, both before and since the publication of the book. It describes how some recent developments have affected the context and perception of the subject and it presents a selection of contemporary applications of Hausdorff measures. These combine to demonstrate the fundamental rôle of Hausdorff measures, both as a subject in their own right and in applications throughout mathematics.

To do justice to the progress that has stemmed from the material in this book since its publication nearly thirty years ago would require far more space than is available. The references cited point to sources of further information but are not necessarily the most significant research publications and not remotely exhaustive. A full bibliography would run to hundreds of pages.

Historical context

Measures, as a device for specifying the size of sets, were essentially introduced by Borel in 1895. In 1915 Carathéodory gave a very general construction for outer measures that included one-dimensional or linear measures as a special case, and he indicated that this could be extended to s-dimensional measures in $\mathbb{R}^n$ for other integers s. In 1919 Hausdorff pointed out, in his famous paper, that such measures could be constructed for *non-integral* s, and amongst various examples he showed that the middle-third Cantor set has positive finite $(\log 2/\log 3)$-dimensional measure. Since then, measures of this type have been referred to as *Hausdorff measures*.

The Hausdorff measure constructed from the *dimension function* or *gauge function* $h \in \mathcal{H}$ is denoted by μ^h (see Chapter 2, §1), with the particularly important case of *s-dimensional Hausdorff measure* μ^{h_s} obtained by taking $h_s(t) = t^s$. The *Hausdorff dimension*, $\dim E$, of a set E is defined to be the infimum value of s such that $\mu^{h_s} < \infty$. (This numerical value of dimension is coarser than that implied by the partial order $\prec$ of Chapter 2 §4.) A set E with $0 < \mu^{h_s}(E) < \infty$ is termed an *s-set*.

A key feature of Hausdorff measures is the geometrical element in their definition which is reflected in their properties. In particular the *scaling property*, that $\mu^{h_s}(\lambda E) = \lambda^s \mu^{h_s}(E)$ where λE is a similar copy of E scaled by a factor λ, generalises the familiar scaling properties of length, area and volume. This geometrical aspect led to the programme, pioneered by Besicovitch and his students through the middle half of the 20th century, of relating the geometry of a set E, in particular properties of its tangents and projections, to properties of Hausdorff measures on E. An early achievement (Besicovitch (1928, 1938)) was to show that a 1-set in the plane could be decomposed into a 'regular' part and an 'irregular' part, which could be distinguished either *measure-theoretically*, in terms of the existence of densities $\lim_{r \to 0}(\mu^{h_s}(B(x,r) \cap E)/r^s)$, or *geometrically*, in terms of rectifiability and the existence of tangents. Later, attention was turned to s-sets for non-integral s; their irregularity is also manifested in both measure-theoretic and in geometric ways (see Marstrand (1954b)).

For much of this period the theory of generalised capacities developed alongside Hausdorff measures, and Frostman [Fro] crystallised the link between potential theory and dimensions in 'Frostman's lemma': that $\mu^{h_s}(E) > 0$ if and only if there exists a (non-trivial) measure ν supported by E such that

$$\nu(B(x,r)) \leq r^s \tag{1}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. It follows that $\dim E$ is the supremum value of s for which there exists a positive measure ν on E such that the energy $\int\int |x-y|^{-s} d\nu(x) d\nu(y)$ is finite. Kaufman's (1968) 'potential theoretic method', which utilises this relationship, has proved an extremely versatile tool in geometrical problems involving projections, intersections, distance sets and Brownian motion, see [Fal1, Fal4, Kah, Mat].

Hausdorff measures and dimensions were found to be very well-suited to quantifying sets of Lebesgue measure zero which were nonetheless 'substantial'. Such sets were encountered in many areas of mathematics, including number theory, stochastic processes and dynamical systems.

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C. A. Rogers
Frontmatter
More information

Besicovitch died in 1970, and his proposed book *Geometry of Sets of Points* on the geometry of Hausdorff measures was never written, though a couple of short draft chapters exist. (Falconer's tract [Fal1] may be something like Besicovitch had in mind.) Self-contained accounts of Hausdorff measures and their geometry appeared in book form for the first time: Federer's *Geometric Measure Theory* [Fed], which includes applications to homological integration theory and the calculus of variations, was published in 1969, and Rogers' *Hausdorff Measures* in 1970. Thus 1970 was a time for taking stock of the progress that had been made with Hausdorff measures and geometric measure theory.

'Hausdorff Measures'

The three chapters of *'Hausdorff Measures'* differ widely in character. Chapter 1 gives an excellent account of basic measure theory, adopting the approach of Carathéodory in assigning an (outer) measure to all subsets, leading to a σ-field of measurable sets. The Method I construction then yields a wide class of measures on general sets. Specialising to measures on topological spaces and then to measures on metric spaces permits the subtle interplay between metric and measure-theoretic properties which is crucial to the geometry of Hausdorff measures. In particular, Method II of constructing measures on metric spaces leads to 'metric measures' with good topological properties, including measurability of Borel and Souslin sets.

The substantial Chapter 2 contains much technical and original material. Hausdorff measures are defined at the outset as a special case of a Method II construction, and many properties follow immediately from Chapter 1. Even if a set has Hausdorff dimension s, its s-dimensional Hausdorff measure may well be infinite or zero, and many interesting results in Chapter 2 relate to this situation. Several theoretical tools are presented, including comparable net measures (which are equivalent to Hausdorff measures but often more manageable) and the increasing sets lemma. The chapter culminates with Theorems 56 and 57 which generalise the work of Besicovitch (1952): under very general conditions, Borel and Souslin sets of positive (perhaps infinite) Hausdorff measure have compact subsets of positive finite measure. For a measure to be a useful tool, it is generally necessary to work within a set that has positive finite measure, and these theorems provide sets that one can 'get one's hands on' in problems involving measure and dimension.

Chapter 3 surveys applications in which Hausdorff measures and dimensions play a major part. The author was very perceptive in his selection: almost all the topics have continued to be active research areas. Two very different applications are presented in greater detail. A concise introduction to continued fractions leads to a dimensional analysis of the set of numbers with all partial quotients 1 or 2. Then classes of continuous increasing functions are represented as integrals with respect to Hausdorff measures, analogously to Lebesgue's theory of differentiation of absolutely continuous functions. The wide variety of topics in this chapter emphasises the scope of Hausdorff measures. Although there have been many advances since the book was written, some of which are outlined below, the chapter remains an invaluable guide to the basic literature.

§2 Recent general developments

Several developments in the years since the publication of *'Hausdorff Measures'* have affected the direction and context of the subject.

The advent of fractals

The area received an enormous boost in the late 1970s from two quarters. Mandelbrot's book *Les Objects fractals: forme, hasard et dimension* [Man2] appeared in 1975 followed by substantially rewritten and expanded versions *Fractals: Form, Chance and Dimension* [Man3] in 1977 and *The Fractal Geometry of Nature* [Man4] in 1982. Mandelbrot's thesis was that 'fractals', as he named highly irregular sets, are the rule rather than the exception both in mathematics and in nature, with the concept of 'fractal (Hausdorff) dimension' central to their study. The scope of these ideas gave a tremendous impetus to mathematicians and scientists to look again at mathematical and natural objects which had hitherto been dismissed by many as too irregular for fruitful study.

At about the same time, computer technology had advanced sufficiently to allow a wide variety of fractals to be drawn easily and accurately. In particular, self-similar fractals, obtained by repeated substitution of a figure within itself, and Julia sets, that is certain sets which are invariant under transformations of the complex plane, attracted renewed attention from mathematicians as well as from those who enjoyed computer experiment. As a natural tool for studying the mathematics and geometry of fractals, Hausdorff dimensions and measures attracted new interest, with several results which had been known fifty years earlier rediscovered as 'new'.

Iterated function systems

In 1981 Hutchinson [Hut] unified many fractal constructions by observing that, given contractions $S_1, \dots, S_m$ on a complete metric space, there exists a unique non-empty compact set E satisfying

$$E = \bigcup_{i=1}^{m} S_i(E); \tag{2}$$

this follows by applying Banach's contraction mapping theorem in an appropriate setting. Such a family of contractions is known as an *iterated function system* (IFS) with E the *attractor* or *invariant set*.

An IFS imposes a natural net structure on its attractor E (see Chapter 2, §7) with net $\mathcal{N} = \{S_{i_1} \circ \cdots \circ S_{i_k}(E) : 1 \leq i_j \leq m\}$. The net measure constructed by Method II from the pre-measure τ on $\mathcal{N}$ given by $\tau(S_{i_1} \circ \cdots \circ S_{i_k}(E)) = \operatorname{diam}(S_{i_1} \circ \cdots \circ S_{i_k}(E))^s$ is often equivalent to s-dimensional Hausdorff measure.

When each S_i is a similarity transformation of ratio r_i and the union in (2) is 'nearly disjoint' (with an '*open set condition*' satisfied) E is called a *self-similar set* and has Hausdorff dimension equal to the value of s satisfying

$$\sum_{i=1}^{m} r_i^s = 1 \tag{3}$$

with $0 < \mu^{h_s}(E) < \infty$. (Dimension formulae of this form were already known to hold for a variety of sets, see Moran (1946).) IFSs provide a succinct representation for large classes of sets which naturally support Hausdorff measures. 'Classical' fractals such as the middle-third Cantor set, the von Koch curve and the Sierpiński triangle fit into this framework.

The IFS approach has been extended to graph-directed systems [MW], systems with infinitely many contractions [MU] and families of random mappings [Fal2, Gra].

The realisation that apparently complicated sets might be represented succinctly by a small number of contractions led to intensive research on 'fractal image compression' with the objective of coding pictures by small data sets to permit fast telephone transmission [Fis, BH].

The thermodynamic formalism

A deep insight of Sinai [Sin], Bowen [Bow] and Ruelle [Rue1] was that techniques from statistical mechanics could be used to study invariant sets of IFSs and dynamical systems involving *non-linear* mappings. In particular, this can give a non-linear version of the dimension formula (3) along with a natural measure equivalent to Hausdorff measure on the invariant set. For a very simple instance, let $S_i : [0,1] \to [0,1]$ ($i = 1, \ldots, m$) be twice differentiable mappings with $0 < c_1 \leq |S_i'(x)| \leq c_2 < 1$ for all $x \in [0,1]$, such that the intervals $S_i[0,1]$ are pairwise disjoint. Let $f : \bigcup_{i=1}^{m} S_i[0,1] \to [0,1]$ be the 'inverse' function defined by $f(x) = S_i^{-1}(x)$ for $x \in S_i[0,1]$. Then the attractor E of the IFS $\{S_1, \ldots, S_m\}$ is also invariant for the expanding mapping f, in the sense that $f(E) = E$. Let I_k denote the set of k-term sequences $I_k = \{(i_1, \ldots, i_k) : 1 \leq i_k \leq m\}$. In estimating the Hausdorff measure of E it is natural to consider the sums over net intervals

$$\textstyle\sum_k^s \equiv \sum_{I_k}(\mathrm{diam}(S_{i_1} \circ \cdots \circ S_{i_k}[0,1]))^s, \tag{4}$$

a sum corresponding to the thermodynamic 'partition function'. It may be shown that the limit $\psi^s = \lim_{k\to\infty} \frac{1}{k} \log \sum_k^s$ exists, and the value of s such that $\psi^s = 0$ equals the Hausdorff dimension of E. (In the case where the S_i are similarities of ratios r_i, this essentially means that $(\sum_{i=1}^m r_i^s)^k$ remains bounded away from 0 and ∞ as $k \to \infty$, which is equivalent to (3).) In fact $\psi^s = P(-s \log |f'|) = 0$ where P is the *pressure functional* defined on functions on E with the parameter s analogous to inverse temperature. The *Gibbs measure*, which occurs naturally in the thermodynamic setting, turns out to be equivalent to s-dimensional Hausdorff measure on E, so $0 < \mu^{h_s}(E) < \infty$. Moreover, this is an *equilibrium measure* that maximises an entropy expression over all invariant probability measures. See [Bar, Bow, Fal6, Rue1, Rue2] for further details.

There are many other classes of IFSs and dynamical systems for which the thermodynamic formalism relates naturally to Hausdorff measures, see below.

Packing measures and dimensions

Another signal development in the early 1980s was the introduction of packing measures and packing dimensions in various forms [ST, Sul, Tri1, TT]. Packing measures are in many ways dual to Hausdorff measures which may be thought of as 'covering' measures. For $E \in \mathbb{R}^n$, $h \in \mathcal{H}$ and $\delta > 0$ let

$$\pi_\delta^h(E) = \sup \sum_i h(\mathrm{diam}(B_i))$$

where the supremum is over all 'δ-packings' of E, that is collections of disjoint balls $\{B_i\}$ of radii at most δ with centres in E. The limit

$$\pi_0^h(E) = \lim_{\delta\to 0} \pi_\delta^h(E)$$

exists, but unfortunately is not in general a Borel measure. However, applying Method II by defining

$$\pi^h(E) = \inf\left\{\sum_i \pi_0^h(E_i) : E \subset \bigcup_{i=1}^{\infty} E_i\right\}$$

yeilds a metric measure, known as *packing measure*. The *packing dimesion*, Dim$E$, of a set E is defined analogously to Hausdorff dimension, as the infimum value of s such that $\pi^{h_s}(E) < \infty$, where $h_s(t) = t^s$, see [Fal4, Mat] for further details.

Many Hausdorff measure properties have packing analogues, and nowadays corresponding Hausdorff and packing results are presented alongside each other. For example, for Borel sets E and F there are symmetrical inequalities for dimensions of products:

$$\dim E + \dim F \leq \dim(E \times F) \leq \mathrm{Dim}(E \times F) \leq \mathrm{Dim}E + \mathrm{Dim}F.$$

It is perhaps surprising that 60 years elapsed between the introduction of Hausdorff measures and packing measures, even given the extra 'Method I' step required in the definition.

Hausdorff-like measures
There are many variants of Hausdorff measure which retain most of the properties of Hausdorff measures discussed in this book. For instance, measures may be constructed by Method II from a pre-measure τ on a class of sets $\mathcal{C}$ (see Theorem 15) where $\tau(C)$ depends on geometrical properties of C other than its diameter. This is the case with Rogers' *dimension print measures* on subsets of $\mathbb{R}^n$: here $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a given non-negative vector and coverings are by rectangular parallelepipeds C with $h(C) = l_1^{\alpha_1} l_2^{\alpha_2} \ldots l_n^{\alpha_n}$, where $l_1, l_2, \ldots, l_n$ are the edge lengths of C in non-increasing order, see [Rog2] and Appendix A. Similarly, a measure constructed using coverings from a net of parallelepipeds is the natural one for work on self-affine sets (that is attractors of IFSs where the mappings S_i in (2) are affine transformations) [Fal3, McM1]. Other Hausdorff-like measures occur in multifractal theory, see below.

Subsets of finite measure
The fact that a Borel or Souslin set of positive (possibly infinite) Hausdorff measure has a compact subset of positive finite measure, is fundamental in the theory and in many applications of Hausdorff measures. This is rightly given prominence in Chapter 2 of this book, where the intricate proofs use comparable net measures and the 'increasing sets lemma'. Recently, a completely different approach was introduced by Howroyd [How, Mat] using *weighted* Hausdorff measures to enable the use of powerful techniques from functional analysis, such as the Hahn–Banach and Krein–Milman theorems.

Frostman's lemma is closely related. Indeed (1) follows easily from the existence of compact subsets of finite positive measure together with the fact that, for an s-set E, the upper density $\limsup_{r\to 0}(\mu^{h_s}(B(x,r) \cap E)/(2r)^s) \leq 1$ for μ^{h_s}-almost all $x \in E$ (which comes from a routine application of the Vitali covering lemma). Not surprisingly, analogous results hold for packing measures: subsets of positive finite packing measure exist under very general conditions [JP]. There is a packing analogue of Frostman's lemma, as well as 'anti-Frostman' lemmas (with the inequalities reversed) for both Hausdorff and packing dimensions, which all go to emphasise the duality of these notions [Cut2, Fal6].

The many applications where one requires a positive finite measure on a given set include the relationships between measures and dimensions of sets E, F and their product $E \times F$, see [How, Mat], and formulae for the dimension of self-affine sets [Fal3].

§3 Recent applications of Hausdorff measures

Professor Rogers claimed that his Chapter 3 'Applications of Hausdorff measures' was 'very inadequate (like a half-hour visit to the British Museum)'. By the same yardstick, this section takes a few minutes to glance at highlights in a museum that has become greatly enlarged and enriched over the past 30 years, though perhaps 'museum' is not quite the right word for an area that remains active and exciting.

Geometrical properties

The programme initiated by Besicovitch's work in 1928 on the geometric structure of 1-sets in the plane still continues, with densities, tangency and rectifiability properties and orthogonal projections studied intensively.

Recall that a set $E \in \mathbb{R}^n$ is *m-rectifiable* if it is made up of countably many pieces which are Lipschitz images of $\mathbb{R}^m$, formally if there exist Lipschitz $f_i : \mathbb{R}^m \to \mathbb{R}^n$ such that $\mu^{h_m}(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m)) = 0$. A (Radon) measure μ on $\mathbb{R}^n$ is *m-rectifiable* if there is an m-rectifiable Borel set E such that $\mu^{h_m}(\mathbb{R}^n \setminus E) = 0$ and μ is absolutely continuous with respect to μ^{h_m}. Work on the relationships between densities and rectifiability of sets and measures culminated in Preiss' proof [Pre] that if the density $\lim_{r\to 0}(\mu(B(x,r))/r^m)$ of a measure μ on $\mathbb{R}^n$ exists and is positive and finite for μ-almost all x then m is an integer and μ is m-rectifiable. In particular, a Borel set E with $\mu^{h_m}(E) < \infty$ is m-rectifiable if and only if $\lim_{r\to 0}(\mu^{h_m}(B(x,r) \cap E)/(2r)^m)$ exists for μ^{h_m}-almost all $x \in E$, in which case the density equals 1. Furthermore there is a number $0 < c(m) < 1$ such that if $\liminf_{r\to 0}(\mu^{h_m}(B(x,r) \cap E)/(2r)^m) \geq c(m)$ for μ^{h_m}-almost all $x \in E$ then E is m-rectifiable. Finding the least possible value of $c(1)$ is an old and intriguing problem: Besicovitch (1928) showed that $1 - 10^{-2576}$ would do, and he later improved this to $\frac{3}{4}$ (Besicovitch (1938)), conjecturing that $\frac{1}{2}$ was least possible. To date the best value that has been established is $\frac{1}{12}(2 + \sqrt{46}) \approx 0.732$ [PT].

There is also a nice packing measure characterisation: if $E \subset \mathbb{R}^n$ with $0 < \pi^{h_s}(E) < \infty$, then $\pi^{h_s}(E) = \mu^{h_s}(E)$ if and only if s is an integer and the restriction of π^{h_s} to E is rectifiable, see [Mat, ST]

Tangent measures have become a major tool in this area. As limits of scaled enlargements of a measure μ about a point, the tangent measures of a measure μ retain certain features of μ but behave in a more regular and tractable way. More precisely, for $a \in \mathbb{R}^n$ and $r > 0$, let $T_{a,r}$ denote the transformation that maps a measure μ to that given by $(T_{a,r}\mu)(A) = \mu(rA + a)$. A non-zero measure ν is a *tangent measure* of μ at a if there are positive sequences $(r_i) \to 0$ and (c_i) such that $c_i T_{a,r}\mu \to \nu$ in the vague topology.

Tangent measures may be used to relate local features of a measure to global properties such as integral dimensionality and rectifiability. Preiss' rectifiability proof makes powerful use of tangent measures to reduce density properties to more manageable questions on the structure of *m-uniform* measures, that is measures ν such that $\nu(B(x,r)) = cr^s$ for all $r > 0$ and all x in sptν.

Tangent measures have been used to show that integral dimensionality and rectifiability follow from even weaker conditions, for example from conditions involving *average densities* such as

$$\liminf_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mu^{h_s}(B(x,e^{-t}))}{(2e^{-t})^s} dt,$$

see [MoP].

Harmonic analysis

Hausdorff measures and geometric measure theory have impacted on harmonic analysis in a variety of ways. In 1919 Besicovitch [Bes] constructed a 'Kakeya' set, that is a set $E \subset \mathbb{R}^2$ of zero plane Lebesgue measure which contains a straight line in every direction, and he later noted (Besicovitch (1964b)) that such sets could be realised by dualising projection properties of Hausdorff measures. The existence of a Kakeya set easily implies that there can be no inequality of the form

$$\int \| F_\theta \|_p \, d\theta \le c \| f \|_{p'}$$

for functions $f : \mathbb{R}^2 \to \mathbb{R}$, where $F_\theta(t) = \int_l f dl$ is the integral of f along l, the line in direction θ and distance t from the origin. In the same vein, much delicate work has been done on maximal functions defined over lower-dimensional manifolds. For example, it is natural to compare $f : \mathbb{R}^2 \to \mathbb{R}$ with the 'circular maximal function' F obtained by setting $F(x)$ equal to the maximum absolute value of the averages of f over all circles with centre x, see [Ste] for an account of this area which has a strong geometrical flavour.

Recently, tangent measures have been used very effectively to relate the existence of singular integrals to rectifiability. For instance, for reasonable measures μ on $\mathbb{R}^n$, if the principal value integral

$$\lim_{\epsilon\to 0} \int_{|y-x|\ge\epsilon} |y-x|^{-s-1}(y-x) d\mu(y)$$

Cambridge University Press
978-0-521-62491-6 - Hausdorff Measures
C. A. Rogers
Frontmatter
More information

exists μ-almost everywhere, then s is an integer and μ is s-rectifiable, see [Mat, MP].

Dimension calculations

Calculating Hausdorff dimensions of specific sets has become something of an industry, with the primary concept of Hausdorff measure often played down. (Is 'A piece of string is one-dimensional' a satisfactory answer to an enquiry about the size of a piece of string?) Nevertheless, a frequent aim is to find $h \in \mathcal{H}$ with $0 < \mu^h(E) < \infty$ if such exists, even if good bounds for $\mu^h(E)$ are too difficult to obtain or unimportant. Formulae and estimates for dimensions and measures have been found for many classes of sets, particularly those that arise as attractors of IFSs (2) where the contractions are of special types. These include self-similar sets (3), self-affine sets [Fal3, McM1], self-conformal sets [Rue2] and statistically self-similar sets [Fal2, GMW]. For references to many other dimension calculations see [Fal4, Fal6, Mat].

Sets defined by number-theoretic properties

Certain sets of numbers defined by conditions on their base d digits have obvious IFS representations. For example, the set of numbers in [0,1] whose decimal digits are all even is the attractor of an IFS consisting of five similarities of ratio $\frac{1}{10}$ and hence by (3) has Hausdorff dimension $\log 5/\log 10$ and positive finite Hausdorff measure. On the other hand, the set E of numbers in $[0,1]$, which when written to base d has a proportion $p_j > 0$ of its digits equal to j (in a limiting sense) for all $j = 0, 1, \ldots, d-1$, is dense in $[0,1]$. By concentrating a measure on E in a natural way and examining densities, E has Hausdorff dimension $s = -(\log d)^{-1} \sum_{j=0}^{d-1} p_j \log p_j$, however in this case $\mu^{h_s}(E)$ is *not* in general positive and finite, see [Bil, Fal4].

The continued fraction sets of Chapter 3, §2 fit naturally in the IFS framework. The set E of numbers in $[0,1]$ with all continued fraction quotients equal to 1 or 2 is the attractor of the IFS consisting of the two (non-linear) contractions $S_1(x) = 1 + 1/x$ and $S_1(x) = 2 + 1/x$, see [Fal4]. Jarník's calculation, that $\frac{1}{3} \le \dim E \le \frac{2}{3}$, has been superseded by methods that enable the dimension to be computed as accurately as desired, with $\dim E = 0.5312805062772051416$ a recent estimate, see [Hen] where dimensions of other sets determined by continued fraction quotients are also listed.

Hausdorff dimension has played a major rôle in the metrical theory of Diophantine approximation. A classical theorem of Jarník states that, for $\alpha > 2$, the set of α-well-approximable numbers has Hausdorff dimension $2/\alpha$. (A number x is *α-well-approximable* if there are infinitely many positive integers q such that $|x - \frac{p}{q}| \le \frac{1}{q^\alpha}$ for some integer p.) There have been many dimension calculations which generalise this. For example, the dimension of the set of points in $\mathbb{R}^n$ for which the coordinates are simultaneously α-well-approximable (with the same denominator occurring in approximations to all coordinates) has been calculated [DRV2], as has the dimension of well-approximable subsets of smooth manifolds [DRV1].

Brownian motion
Dimension and measure properties of Brownian motion are discussed in Chapter 3, §1 and this is another active area. Whilst the sample path B of Brownian motion in $\mathbb{R}^n$ ($n \geq 2$) has Hausdorff dimension 2, this is an instance where the 'correct' dimension function $h \in \mathcal{H}$ involves a logarithmic term: with probability one, $0 < \mu^h(B) < \infty$ where $h(t) = t^2 \log(t^{-1}) \log\log\log(t^{-1})$ if $n = 2$ and $h(t) = t^2 \log\log(t^{-1})$ if $n = 3, 4, \ldots$. Calculations have been extended to Brownian functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, to fractional Brownian motion (where the distance increments over time intervals t have variance $t^{2\alpha}$ for $0 < \alpha < 1$) and to general stable processes. Results on double points and multiple points of sample paths of these processes may be deduced from general properties of dimensions of intersections of sets in general position, see [Fal4, Kah]. Studies have been made of the dimensions of zero sets and level sets of processes, of 'slow points' and 'fast points', and of the images of given sets under such processes. Particularly important is the identification of local times of Brownian and other processes with an appropriate Hausdorff measure. There are packing measure analogues: in contrast to Hausdorff measure, the dimension function $h(t) = t^2/\log\log(t^{-1})$ is the one that assigns positive finite packing measure to Brownian paths in $\mathbb{R}^n$ for $n = 3, 4, \ldots$ [TT]. For surveys of this vast subject area see [Adl, Kah, Tay].

Two random sets B and E are *intersection equivalent* in a bounded region U if for every closed set $K \subset U$,

$$a \leq \frac{\mathrm{P}(K \cap B \neq \emptyset)}{\mathrm{P}(K \cap E \neq \emptyset)} \leq b,$$

where P denotes probability and a, b are positive constants independent of K. Let $E_n(p)$ be the statistically self-similar set obtained from the unit cube in $\mathbb{R}^n$ by repeated subdivision of cubes into 2^n subcubes, with each subcube independently retained with probability p. Then the Brownian sample path B in $\mathbb{R}^n$ ($n \geq 3$) is intersection equivalent to $E_n(2^{2-n})$, see [Per]. This remarkable correspondence is a consequence of connections between Brownian motion, potential theory and branching processes. It enables results, for example on dimensions and on multiple points of Brownian paths, to be deduced easily from straightforward properties of $E_n(p)$. There are similar intersection equivalences for other stable processes.

Local dimensions and multifractal theory

For μ a Borel regular measure on $\mathbb{R}^n$ with $0 < \mu(\mathbb{R}^n) < \infty$, the *(lower) local* or *pointwise dimension* of μ at x is defined by

$$\underline{\dim}_{\mathrm{loc}}\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

For certain 'rich' measures μ the sets

$$E_\alpha = \{x : \underline{\dim}_{\mathrm{loc}}\mu(x) = \alpha\}$$

may be 'large' for a range of α. There are two natural ways in which to quantify E_α: using the original measure μ and by its Hausdorff dimension. The former approach was adopted by Rogers and Taylor (1959, 1962) who, roughly speaking, decomposed μ into component measures with local dimension α, over a range of α. More recent treatments are given in [Cut1, KK].

The second approach considers the dimension $\dim E_\alpha$, which may be significant even if $\mu(E_\alpha) = 0$. This is the basis of *multifractal* theory, an area of intense activity since the mid-1980s. The multifractal concept may be traced back to Mandelbrot's work on turbulence [Man1] in 1974 in which he suggested that dissipation of energy in a turbulent fluid is concentrated on a fractal, with different moments scaling at different rates, an idea that was developed in the physics literature, see for example [FP, HJKPS, HP]. A basic aim is to find the *multifractal spectrum* or *singularity spectrum* $\dim E_\alpha$ of a measure μ, which may be non-zero over a range of α.

Many natural questions about multifractals parallel those about fractals. Spectra have been calculated in specific instances, such as for 'self-similar' measures [CM] and 'self-affine' measures [Kin]. Geometrical properties of multifractals, for example concerning projections [HK] and products [Ols2], have also been investigated. It is not surprising that, just as Hausdorff measures are used in the analysis of fractal sets, so Hausdorff-like measures have been invoked to study multifractals [BMP, Ols1, Pes]. Very roughly, for $q, \beta \in \mathbb{R}$, setting $h(G) = \mu(G)^q \mathrm{diam}(G)^\beta$ in Definition 16 of Chapter 2 leads to a Hausdorff-like measure $\mu^{q,\beta}$ (not strictly a Hausdorff measure since $h(G)$ does not depend solely on the diameter of G). Analogously to Hausdorff dimension, for each q let $\beta(q)$ be the infimum value of β for which $\mu^{q,\beta}(\mathrm{spt}(\mu))$ is finite. For many measures, for example self-similar measures or statistically self-similar measures, the *multifractal formalism* applies: the multifractal spectrum is the Legendre transform of $\beta(q)$, that is

$$\dim E_\alpha = \inf_{-\infty<q<\infty} \{\beta(q) + \alpha q\}.$$

For fuller accounts see [EM, Fal6, Ol1].

A parallel theory of 'multifractal analysis of functions' quantifies sets where the local Hölder exponent of a function takes given values, see [Jaf].

Dynamical systems

Dynamical systems have been researched intensively over the past 30 years, with computer experiment again a motivating influence. A (discrete) dynamical system is essentially a map $f : D \to D$ (where the domain D is often a subset of $\mathbb{R}^n$), with the k-th iterate $f^k(x)$ thought of as the position at time k of a particle initially at x. A set $E \subset D$ is an *attractor* of f if E is a closed invariant subset of D (so $f(E) = E$) such that $\operatorname{dist}(f^k(x), E) \to 0$ as $k \to \infty$ for all initial x in some open *basin of attraction* containing E. Similarly, an invariant set E is a *repeller* if points near E move away under iteration. Even very simple systems can have fractal attractors or repellers, often supporting a Hausdorff measure as a natural invariant measure.

The set E is *hyperbolic* for the map f (which we take to be twice differentiable) if, very roughly, the absolute values of the eigenvalues of the derivative of f on E are bounded away from 1, so that $f'(x)$ splits into an expanding and a contracting part. Then E has a *Markov partition*, that is a decomposition into a finite number of components on which the branches of the inverse f^{-1} behave very like a (graph-directed) IFS, see [KH]. This is a situation where the thermodynamic formalism may be applied, often leading to a 'pressure formula' for the dimension of E and with Hausdorff measures arising naturally as Gibbs measures. Dynamical systems that have been analysed in this way include expanding conformal mappings [Bow, Rue2], cylinder maps [Bed], general expanding mappings [Fal5], and hyperbolic mappings with both expanding and contracting directions [MM].

The dynamics of conformal mappings on $\mathbb{C}$ remains a prolific research area. Although Fatou [Fat] and Julia [Jul] developed a substantial theory of the dynamics of polynomial mappings $f : \mathbb{C} \to \mathbb{C}$ in 1918, it was not until relatively recently that it was realised that the repellers or 'Julia sets' that arise are in general fractals. In particular, the dynamics of the quadratic mappings $f_c(z) = z^2 + c$ and the structure of the Julia set vary dramatically according to the location of c relative to the *Mandelbrot set* M (the set of c for which f_c has a connected Julia set, equivalently those c such that $f_c^k(0) \not\to \infty$). For c outside M and for c in many interior regions of M, the mapping f_c is hyperbolic, making the dynamics tractable and permitting the thermodynamic formalism. However, for certain c on the boundary of M there are still unanswered questions, see [Dev, McM2] for further details.

Fuchsian and Kleinian groups

Two-dimensional hyperbolic space may be represented by the open unit disc D in $\mathbb{C}$ with appropriately defined hyperbolic metric d. The conformal isometries with respect to d are the Möbius transformations $g(z) = (az+\overline{c})/(cz+\overline{a})$ where $a, c \in \mathbb{C}$ satisfy $|a|^2 - |c|^2 = 1$. A group G of Möbius transformations on D is called a *Fuchsian group* if it acts discontinuously on D, that is for every compact $K \subset D$ the intersection $g(K) \cap K$ is non-empty for only finitely many $g \in G$. The *limit set* L of G consists of the points ζ on the boundary of D for which there is a sequence $g_n \in G$ with $g_n(0) \to \zeta$. If G is cyclic or is an extension of a cyclic group by an order 2 element then L contains at most two points. However, for other Fuchsian groups L has a fractal structure.

The limit set L carries a natural and very useful measure. The infimum of s for which the *Poincaré series* $S(s) = \sum_{g\in G} e^{-sd(g(0),0)}$ converges is termed the *critical exponent* s_0. Very roughly, the *Patterson measure* μ is the measure on L given by the weak limit as $s \searrow s_0$ of the normalised measures

$$\mu_s = S(s)^{-1} \sum_{g\in G} e^{-sd(g(0),0)} \delta_{g(0)},$$

where δ_z denotes the unit point mass at z. (Provided $S(s_0)$ diverges it is not difficult to see that the limit exists and is supported by L; otherwise it is necessary to introduce weightings in the limiting process.)

The Patterson measure has many nice invariance properties and is the natural measure with which to study the limit set. For wide classes of groups G the critical exponent s_0 equals the Hausdorff dimension of L, and often μ turns out to be equivalent to s_0-dimensional Hausdorff and/or packing measure on L (in some instances Hausdorff and packing measures restricted to L are not equivalent), see [Sul].

An analogous theory goes through for *Kleinian groups*, that is groups of isometries on higher-dimensional hyperbolic spaces.

Manifolds of constant negative curvature arise as quotients of hyperbolic space by Kleinian groups, and geodesics on such manifolds may be identified with pairs of points on the boundary of D. Patterson measures, and hence geometric measures, are central in studying dynamics on these manifolds and lead to estimates for the dimensions of certain sets of geodesics corresponding to points in the limit set, see [Bea, Nic, Pat].

Differential equations

Some very interesting recent developments concern differential equations.

Hausdorff measures are defined on general metric spaces (see Chapter 2), so there is no problem in working with measures and dimensions of sets in infinite-dimensional normed spaces, such as spaces of functions. This is an appropriate setting for studying certain differential equations, for instance the reaction–diffusion equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + p(u)$$

for an appropriate non-linear function p. Consider the evolution of solutions $u(x,t)$ with time $t \geq 0$ where $x \in D$ for some spatial domain D. The solution $u(.\,,t)$ at time t may be regarded as a point in some normed subspace X of the space of continuous functions on D, with $u(.\,,t) \equiv f_t(u(.\,,0))$, where the evolution operator f_t maps the initial condition $u(.\,,0)$ to the solution at time t. The *attractor* of the differential equation may be defined as the maximal compact set E of functions in X that is invariant under f_t (so $f_t(E) = E$) for all $t > 0$ such that the $\operatorname{dist}(f_t(u(.\,,0)), E) \to 0$ as $t \to \infty$ for all initial $u(.\,,0) \in X$. Careful estimates lead to upper bounds for the Hausdorff dimension of the attractor in terms of parameters of the equation. This attractor in function space represents the permanent regime that can be observed when the system starts from any initial conditions. Its Hausdorff dimension indicates the complexity of the flow and may be thought of as the number of degrees of freedom of the system.

This type of analysis has been applied to a wide variety of differential equations, such as the Navier–Stokes equation, pattern formation equations and non-linear Schrödinger equations, see [Lad, Tem].

Various familiar differential equations have been studied in a 'fractal setting'. For example, the dominant term of the asymptotic distribution of the eigenvalues of the Laplacian on an open domain D depends on the area or volume of the domain, but the second-order term often reflects the fractality of the boundary of D [Lap]. Similarly, for the heat equation on a region where the boundary is held at a fixed temperature, the rate of heat loss across the boundary is related to a dimension of the boundary [Ber, FLV].

There is considerable interest in problems where the domain itself is fractal, for example modelling heat diffusion on a Sierpiński triangle. A major difficulty is how to define operators such as the Laplacian in this context; the usual approach is as a limit of difference operators on discrete graphs which approximate the domain, see [BP, Kig].

§4 Further reading

There are several accounts of Hausdorff measures and geometric measure theory which provide further details of the areas sketched above as well as of related topics. These include the books by Edgar [Edg1], Falconer

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C. A. Rogers
Frontmatter
More information

[Fal1] and Mattila [Mat], and Wicks [Wic] whose approach uses non-standard analysis. The books by Barnsley [Barn], Falconer [Fal4,Fal6] and Peitgen, Jürgens and Saupe [PJS] are more concerned with analysis of dimensions and fractals, Massopust [Mas] concentrates on functions and surfaces, and Tricot [Tri2] on curves. The anthology by Edgar [Edg2] provides an interesting historical pespective. The conference proceedings [BD, BGZ, BKS] also contain many relevant articles.

Pertinent areas which have not been touched on here include harmonic measures on fractals [JM], wavelet methods [Hol], other definitions of dimension [Fal4, Man4], minimal surfaces [Mor], and geometric integration theory [HN].

St Andrews, November 1997

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PREFACE

Measures are of importance in mathematics in two rather different ways. Measures can be used for estimating the size of sets, and measures can be used to define integrals. E. Borel in his 1894 thesis (see Borel 1895 or 1940)† essentially introduced the Lebesgue outer measure as a means of estimating the size of certain sets, so that he could construct certain pathological functions. Lebesgue (1904) on the other hand was mainly interested in his measures as a tool enabling him to construct his integral. While both aspects of measure theory are important, the emphasis in this book will be almost entirely on the first; we only mention the theory of integration in the last section of the last chapter.

The first 'Hausdorff' measure was introduced by C. Carathéodory (1914), in a paper in which he also introduced the much more general Carathéodory outer measures. Carathéodory developed the theory of linear measure in n-dimensional Euclidean space and in a final paragraph clearly showed how p-dimensional measure could be introduced in q-dimensional space, for $p = 1, 2, \ldots, q$. The p-dimensional measures, for general positive real p, were introduced by F. Hausdorff (1919); he also illustrated the use of these measures by showing that the Cantor ternary set has in a certain sense the fractional dimension $\log 2/\log 3 = 0{\cdot}6309 \ldots$. The theory of Hausdorff measures has developed very greatly since 1919, very largely as a result of the work of A. S. Besicovitch and his students.

This book cannot contain an account of more than a tiny fraction of the work that has been done on Hausdorff measures. After a first chapter giving an introduction to measure theory with special attention to the study of non-σ-finite measures, the second chapter develops the most general aspects of the theory of Hausdorff measures, and the final chapter gives an account of the applications of Hausdorff measures, a general survey being followed by detailed accounts of two rather special topics.

Much of this book is based on postgraduate lectures given at the University of British Columbia and at University College London. I am conscious that I have been considerably influenced by close and enjoyable contacts with Maurice Sion, when I visited the University

† Such references are to the bibliography, p. 169.

of British Columbia, and with Roy O. Davies, when he visited University College London. Indeed I must hasten to acknowledge that much of §7 of Chapter 2 is the result of joint work with Dr Davies, and that I am most grateful to him for permission to publish it first in this work. Dr Davies has also helped me by careful criticism of Chapter 1. As the book is largely based on lectures, and, as I like my students to follow my lectures, proofs are given in great detail; this may bore the mature mathematician, but it will I believe be a great help to anyone trying to learn the subject *ab initio*.

C. A. ROGERS

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I am most grateful to the Cambridge University Press for the care they have taken in the production of this book, and to Miss S. Burrough for the assistance she has given with the proof-reading.

C. A. R.

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C. A. R.