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MEASURES IN ABSTRACT, TOPOLOGICAL AND METRIC SPACES

§1 Introduction

In his fundamental book H. Lebesgue (1904) introduces an exterior measure or outer measure $m_e(E)$ and an interior measure or inner measure $m_i(E)$ associated with *every* subset E of the real line. He then associates a measure $m(E)$, the common value of $m_e(E)$ and $m_i(E)$, with those sets for which $m_e(E) = m_i(E)$. When more general measures were studied by J. Radon (1913) and C. Carathéodory (1914), these authors adopted slightly different points of view. Radon placed his emphasis on the measures of countably additive set functions defined on a Borel ring of sets and Carathéodory concentrated his attention on the outer measure defined for all sets. In this chapter we adopt the Carathéodory point of view, in this, as in many other ways; but we take care to explain the relationship between the two points of view and to show that they are essentially equivalent.

We make no attempt to present a complete account of any aspect of measure theory, giving only an introduction developing those parts of the general theory that are useful for the study and appreciation of Hausdorff measures. In view of this purpose we take care to prove, in as far as this is possible, results that apply to non-σ-finite measures.

§2 Measures in abstract spaces

In this section we develop the theory of measures in abstract spaces from the 'outer measure' point of view. We first define measures and give a number of examples. We then introduce the concept of measurability and develop the elementary properties of measurable sets. We then introduce 'Method I', a method of defining a measure from a set function satisfying only the weakest of conditions, and show that every measure can be regarded as a measure constructed in this way. We then establish the relationships between the 'outer measure' approach to measure theory and the equally valid approach through measures defined on σ-fields of sets. This leads us to a study of regular measures. Finally, we give methods of obtaining a measure by relativizing a given measure and by taking the supremum of a family of measures.

As soon as one agrees to call the elements of a set 'the points' of the set, the set becomes an abstract space. In this section we shall work with such a space Ω. We introduce

***Definition* 1.** *A function μ defined on the sets of a space Ω is called a measure on Ω if it satisfies the conditions*:

(*a*) *$\mu(E)$ is a non-negative real number or $+\infty$ for each sub-set E of Ω*;

(*b*) $\mu(\varnothing) = 0$;

(*c*) *if $E_1 \subset E_2$ then $\mu(E_1) \leqslant \mu(E_2)$; and*

(*d*) *if $\{E_i\}$ is any sequence of sets of Ω then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} \mu(E_i).$$

Here we depart from standard practice; most authors call the set functions satisfying these conditions 'outer measures' and reserve the name 'measure' for the restriction of such an 'outer measure' to a σ-field (see definition 3 below) of sets on which it is countably additive (see definition 4 below).

Even in a completely abstract space it is easy to give examples of such measures.

Example A. If E is a finite set of points take $\mu(E)$ to be the number of points of E, if E is an infinite set of points take $\mu(E) = +\infty$. We call this measure 'counting measure'.

Example B. Choose a fixed infinite cardinal and take $\mu(E) = 0$, if the cardinal of E does not exceed the fixed cardinal, and take

$$\mu(E) = +\infty,$$

if the cardinal of E does exceed the fixed cardinal.

The second example illustrates a general method of passing from a suitable set property to a measure. Suppose we have a set property S that satisfies the conditions:

(*a*) $\varnothing$ has S;

(*b*) if E_1 has S and $E_2 \subset E_1$ then E_2 has S;

(*c*) if $\{E_i\}$ is a sequence of sets having S then $\bigcup_{i=1}^{\infty} E_i$ has S.

Here the property S can be read as meaning that E is small, in some sense, if E has S. We get examples of measures by taking such a property.

Example C. Let S be a set property satisfying (a), (b) and (c). Take $\mu(E) = 0$, if E has S, and $\mu(E) = 1$, if E has not got S.

Example D. Let S be a set property satisfying (a), (b) and (c). Take $\mu(E) = 0$, if E has S, and $\mu(E) = +\infty$, if E has not got S.

It is easy to give examples of set properties satisfying (a), (b) and (c).

(i) E is at most countable.

(ii) E has cardinal not exceeding some fixed infinite cardinal.

(iii) When Ω is a topological space; E is a countable union of nowhere dense sets (i.e. E is of the first category).

(iv) When Ω is a Hausdorff space; E is a subset of some countable union of compact sets.

(v) When Ω is a topological space and n is given; E is a countable union of sets of topological dimension at most n, for any reasonable definition of topological dimension.

(vi) When a measure ν is given, E has $\nu(E) = 0$.

(vii) When a measure ν is given, E is a countable union of sets of finite ν measure.

Coming down to n-dimensional Euclidean space R_n we have the well-known examples.

Example E. When Ω is R_n; let $\mu(E)$ be the Lebesgue (outer) measure of E (see §5 below).

Example F. When Ω is R_n; let $\mu(E)$ be the upper Lebesgue–Stieltjes or Radon integral

$$\overline{\int_E} dF(x),$$

where F is a non-negative interval function (see Radon, 1913).

2.1. While our measures are defined on all the sets of the space they have rather special properties on the class of 'measurable' sets. We work with a variant of Carathéodory's definition.

***Definition* 2.** *If μ is a measure on Ω, a set E is said to be μ-measurable if for all sets A, B with*

$$A \subset E, \quad B \subset \Omega \backslash E \tag{1}$$

we have

$$\mu(A \cup B) = \mu(A) + \mu(B). \tag{2}$$

Here we use the symbol '\' between two sets to mean 'set theoretic difference'; $A \backslash B$ meaning the set of those points of A not in B (this can be read 'A less B').

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We say that sets A, B satisfying (1) are separated by E; the definition takes the form: *E is μ-measurable if μ is additive on sets that are separated by E.*

As we shall often have to check the measurability of sets it is convenient to have a measurability criterion that is slightly easier to use.

***Theorem* 1.** *If μ is a measure on Ω, then a set E is μ-measurable, if we have*

$$\mu(A \cup B) \geqslant \mu(A) + \mu(B), \tag{3}$$

whenever A and B are sets of finite μ-measure that are separated by E.

Proof. Applying the defining property (d) of a measure to the sequence $A, B, \varnothing, \varnothing, \ldots$ and using the property (b) we have

$$\mu(A \cup B) \leqslant \mu(A) + \mu(B)$$

for all sets A, B. So, to prove the equality (2) for all sets A, B separated by E, it suffices to prove that the inequality (3) holds whenever one of the sets A, B separated by E has infinite measure. But in this case the defining property (c) ensures that both sides of (2) have the value $+\infty$. Hence E is μ-measurable.

***Corollary*.** *If the only values taken by the measure μ are 0 or ∞, all sets are μ-measurable.*

Proof. Whenever A and B have finite μ-measure

$$\mu(A \cup B) \geqslant 0 = \mu(A) + \mu(B).$$

2.2. We are now in a position to state a theorem giving considerable information about the measurable sets and the behaviour of the measure on the measurable sets.

***Theorem* 2.** *Let μ be a measure on Ω. Then*

(a) *if $\mu(N) = 0$, N is μ-measurable;*

(b) *if E is μ-measurable, so is $\Omega \backslash E$;*

(c) *if $\{E_i\}$ is a sequence of μ-measurable sets, $\bigcup_{i=1}^{i=\infty} E_i$ and $\bigcap_{i=1}^{i=\infty} E_i$ are μ-measurable;*

(d) *if $\{E_i\}$ is a disjoint sequence of μ-measurable sets,*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

Before we prove this theorem it will be convenient to introduce the concept of a σ-field† and to prove a simple lemma on σ-fields.

† Read 'sigma-field'.

***Definition* 3.** *A system $\mathscr{A}$ of sets is called a σ-field if it has the three properties*:

(*a*) $\varnothing \in \mathscr{A}$;
(*b*) *if* $A \in \mathscr{A}$, *then* $\Omega \backslash A \in \mathscr{A}$;
(*c*) *if* $A_i \in \mathscr{A}$ *for* $i = 1, 2, \ldots$, *then* $\bigcup_{i=1}^{i=\infty} A_i \in \mathscr{A}$.

***Lemma* 1.** *Let $\mathscr{A}$ be a system of sets with the four properties*:

(*a*) $\varnothing \in \mathscr{A}$;
(*b*) *if* $A \in \mathscr{A}$, *then* $\Omega \backslash A \in \mathscr{A}$;
(c_1) *if* $A_1 \in \mathscr{A}$ *and* $A_2 \in \mathscr{A}$ *then* $A_1 \cup A_2 \in \mathscr{A}$;
(c_2) *if* $A_i \in \mathscr{A}$ *for* $i = 1, 2, \ldots$, *and* $A_1, A_2, \ldots$, *are disjoint, then* $\bigcup_{i=1}^{i=\infty} A_i \in \mathscr{A}$.

Then $\mathscr{A}$ is a σ-field.

Proof. By conditions (*a*) and (*b*) the corresponding conditions of definition 3 hold. So we have only to establish condition (*c*) of the definition. Let $A_1, A_2, \ldots$ be any sequence of sets of $\mathscr{A}$. Then we can write

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} [A_i \cap \{\Omega \backslash \bigcup_{j<i} A_j\}]. \tag{4}$$

By condition (*a*) when $i = 1$, and trivially when $i = 2$ and by condition (c_1), iterated when necessary, when $i \geqslant 3$, the sets

$$\bigcup_{j<i} A_j \quad (i = 1, 2, \ldots)$$

all belong to $\mathscr{A}$. Using (*b*) and (c_1) it follows that the sets

$$A_i \cap \{\Omega \backslash \bigcup_{j<i} A_j\} = \Omega \backslash [\{\Omega \backslash A_i\} \cup \{\bigcup_{j<i} A_j\}]$$

belong to $\mathscr{A}$. As these sets are disjoint sets of $\mathscr{A}$, it follows from (c_2) and (4) that $\bigcup_{i=1}^{i=\infty} A_i$ belongs to $\mathscr{A}$, as required.

Remark. The formula

$$\bigcap_{i=1}^{\infty} A_i = \Omega \backslash \left[\bigcup_{i=1}^{\infty} \{\Omega \backslash A_i\} \right]$$

shows that a σ-field is automatically closed under the operation of countable intersection.

Proof of theorem 2. The proof will be given in parts.

Proof of (*a*). Suppose that N is a set with $\mu(N) = 0$ and that A, B are separated by N with $A \subset N$, $B \subset \Omega \backslash N$. Then using the defining properties in the order (*c*), (*d*), (*c*) and (*b*), we obtain

$$\begin{aligned}\mu(B) \leqslant \mu(A \cup B) &\leqslant \mu(A)+\mu(B)\\ &\leqslant \mu(N)+\mu(B)\\ &= \mu(B).\end{aligned}$$

Hence we must have equality throughout and N is μ-measurable.

Proof of (*b*). Let E be μ-measurable. Then $\Omega\backslash E$ is also μ-measurable by the symmetry of the definition in E and $\Omega\backslash E$.

Proof of (c) *for the union of two sets.* Let E_1, E_2 be two μ-measurable sets. Let A and B be any two sets with finite μ-measure and with

$$A \subset E_1 \cup E_2, \quad B \subset \Omega\backslash(E_1 \cup E_2).$$

Now $$A \cup B = [A \cap E_1] \cup [\{A \cup B\} \cap \{\Omega\backslash E_1\}]$$

and the sets $A \cap E_1$ and $\{A \cup B\} \cap \{\Omega\backslash E_1\}$ are separated by the measurable set E_1. Hence

$$\mu(A \cup B) = \mu(A \cap E_1)+\mu(\{A \cup B\} \cap \{\Omega\backslash E_1\}). \tag{5}$$

But $$\{A \cup B\} \cap \{\Omega\backslash E_1\} = \{A \cap (\Omega\backslash E_1)\} \cup B$$

and the sets $A \cap (\Omega\backslash E_1)$ and B are separated by the measurable set E_2. Hence

$$\mu(\{A \cup B\} \cap \{\Omega\backslash E_1\}) = \mu(A \cap (\Omega\backslash E_1))+\mu(B). \tag{6}$$

Finally, as E_1 is measurable

$$\mu(A \cup E_1)+\mu(A \cap (\Omega\backslash E_1)) = \mu(A). \tag{7}$$

So, by (5), (6) and (7) we have

$$\mu(A \cup B) = \mu(A)+\mu(B)$$

and $E_1 \cup E_2$ is μ-measurable.

Proof of (c) *for the union of a disjoint sequence of μ-measurable sets and proof of* (d). *Let* $\{E_i\}$ be a disjoint sequence of μ-measurable sets. Write

$$E = \bigcup_{i=1}^{\infty} E_i$$

and let A and B be sets with

$$A \subset E \quad \text{and} \quad B \subset \Omega\backslash E.$$

By repeated application of the result of the last paragraph, the set $\bigcup_{i=1}^{i=n} E_i$ is μ-measurable for each positive integer n. Hence

$$\begin{aligned}\mu(A \cup B) &\geqslant \mu\left(\left[A \cap \left\{\bigcup_{i=1}^{n} E_i\right\}\right] \cup B\right)\\ &= \mu\left(A \cap \left\{\bigcup_{i=1}^{n} E_i\right\}\right)+\mu(B),\end{aligned} \tag{8}$$

as
$$B \subset \Omega \backslash E \subset \Omega \backslash \left\{ \bigcup_{i=1}^{n} E_i \right\}.$$

Since the sets $E_n, E_{n-1}, \ldots, E_1$ are disjoint and μ-measurable, we have

$$\begin{aligned} \mu\left(A \cap \left\{ \bigcup_{i=1}^{n} E_i \right\}\right) &= \mu\left(\left[A \cap \left\{ \bigcup_{i=1}^{n-1} E_i \right\}\right] \cup [A \cap E_n]\right) \\ &= \mu\left(A \cap \left\{ \bigcup_{i=1}^{n-1} E_i \right\}\right) + \mu(A \cap E_n) \\ &= \mu\left(A \cap \left\{ \bigcup_{i=1}^{n-2} E_i \right\}\right) + \mu(A \cap E_{n-1}) + \mu(A \cap E_n) \\ &= \ldots \\ &= \mu(A \cap E_1) + \mu(A \cap E_2) + \ldots + \mu(A \cap E_n). \end{aligned}$$

Using this in (8), we obtain

$$\mu(A \cup B) \geqslant \sum_{i=1}^{n} \mu(A \cap E_i) + \mu(B).$$

As this holds for each integer n, it implies that

$$\mu(A \cup B) \geqslant \sum_{i=1}^{\infty} \mu(A \cup E_i) + \mu(B).$$

So, using the defining property (d) of a measure,

$$\begin{aligned} \mu(A \cup B) &\geqslant \sum_{i=1}^{\infty} \mu(A \cap E_i) + \mu(B) \\ &\geqslant \mu\left(A \cap \left\{ \bigcup_{i=1}^{\infty} E_i \right\}\right) + \mu(B) \\ &= \mu(A) + \mu(B) \\ &\geqslant \mu(A \cup B). \end{aligned} \tag{9}$$

In the first place, this shows that $\mu(A \cup B) \geqslant \mu(A) + \mu(B)$ so that $\bigcup_{i=1}^{i=\infty} E_i$ is μ-measurable. Secondly, taking the special case when $B = \varnothing$, and noting that equality must hold throughout (9), we obtain

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A \cap E_i), \tag{10}$$

for all subsets A of $E = \bigcup_{i=1}^{i=\infty} E_i$. Thirdly, taking $A = E$ in (10) we obtain

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i). \tag{11}$$

Thus we have established (d) and also (c) for the union of a disjoint sequence of μ-measurable sets.

Proof of (c). By (a), the set $\varnothing$ is μ-measurable. Further we have already shown that the system of μ-measurable sets is closed under the operations of complementation with respect to Ω, of union of two sets, and a countable disjoint union. It follows, by Lemma 1 that this system is a σ-field and so is closed under the operations of countable union and intersection. This completes the proof of the theorem.

Corollary. *Let T be any set in Ω. Let $\{E_i\}$ be a disjoint sequence of μ-measurable sets. Then*

$$\mu\left(T \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(T \cap E_i). \tag{12}$$

Proof. Applying (10) with

$$A = T \cap \bigcup_{i=1}^{\infty} E_i,$$

we obtain

$$\mu\left(T \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu\left(T \cap \left\{\bigcup_{j=1}^{\infty} E_j\right\} \cap E_j\right) = \sum_{i=1}^{\infty} \mu(T \cap E_i),$$

as required.

2.3. At this stage it is convenient to introduce the concept of a countably additive measure defined on a σ-field of sets.

Definition 4. *A set function μ defined on a σ-field $\mathscr{A}$ of sets is called a countably additive measure on $\mathscr{A}$ if it has the following properties:*

(a) $0 \leqslant \mu(A) \leqslant +\infty$ *for all A in $\mathscr{A}$;*

(b) $\mu(\varnothing) = 0$;

(c) *whenever $\{A_i\}$ is a disjoint sequence of sets of $\mathscr{A}$,*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

Remark. Many authors reserve the name 'measure' for such countably additive measures defined on a σ-field of sets.

In terms of this definition we can reword Theorem 2 to yield

Theorem 3. *If μ is a measure on Ω, the system $\mathscr{M}$ of μ-measurable sets is a σ-field containing the null sets (i.e. those sets N with $\mu(N) = 0$), and the restriction of μ to $\mathscr{M}$ is a countably additive measure on $\mathscr{M}$.*

2.4. The relationship between the measure μ defined on all the subsets of Ω and its restriction to its measurable sets is close. Given a countably additive measure ν defined on a σ-field $\mathscr{A}$ we can, by a process that we shall shortly describe, extend ν to yield a measure λ defined on all sets of Ω, with

$$\lambda(A) = \nu(A) \quad \text{for} \quad A \in \mathscr{A},$$

in such a way that $\mathscr{A}$ is a subfield of the σ-field of λ-measurable sets. In the special case, when ν is the restriction of a measure μ defined on Ω to the μ-measurable sets, and μ is a regular measure (see Theorem 7 below), the measure λ generated from ν will coincide with μ.

For this particular purpose we need a method of constructing a measure from a countably additive measure defined on a σ-field. For other reasons it is desirable to have a much more general method of constructing a measure from a 'pre-measure' defined on any class of sets that contain $\varnothing$. Following Munroe (1953) we shall call this method 'Method I'.

***Definition* 5.** *A function τ defined on a class $\mathscr{C}$ of subsets of Ω will be called a pre-measure, if:*

(*a*) $\varnothing \in \mathscr{C}$;
(*b*) $0 \leqslant \tau(C) \leqslant +\infty$ *for all C in $\mathscr{C}$*;
(*c*) $\tau(\varnothing) = 0$.

***Theorem* 4.** *If τ is a pre-measure defined on a class $\mathscr{C}$ of sets, the set function*

$$\mu(E) = \inf_{\substack{C_i \in \mathscr{C} \\ \cup C_i \supset E}} \sum_{i=1}^{\infty} \tau(C_i) \tag{13}$$

is a measure on Ω.

Remarks. We adopt the convention that any infimum taken over an empty set of real numbers has the value $+\infty$. When no confusion can arise we will abbreviate the formula (13) to the form

$$\mu(E) = \inf_{\cup C \supset E} \Sigma\, \tau(C_i). \tag{14}$$

We shall call the measure μ the measure constructed from the pre-measure τ by Method I. We can call a sequence $\{\mathscr{C}_i\}$ of sets of $\mathscr{C}$ with $\bigcup_{i=1}^{i=\infty} C_i \supset E$ a covering of E with sets from $\mathscr{C}$ and we can call the sum $\sum_{i=1}^{i=\infty} \tau(C_i)$ the τ-value of the covering. Then $\mu(E)$ is the infimum of the τ-values of the coverings of E by sets from $\mathscr{C}$. Our notation conventionally excludes the possibility of using a finite system of sets

$$C_1, C_2, \ldots, C_n$$

of $\mathscr{C}$ covering E; but given such a covering we can replace it by the covering $\{D_i\}$ with

$$D_i = C_i \quad (i = 1, 2, \ldots, n),$$

$$D_i = \varnothing \quad (i = n+1, n+2, \ldots),$$

and we then have

$$\sum_{i=1}^{\infty} \tau(D_i) = \sum_{i=1}^{n} \tau(C_i).$$

Proof. (*a*) As $0 \leqslant \tau(C) \leqslant +\infty$ for all C in $\mathscr{C}$, we clearly have

$$0 \leqslant \mu(E) \leqslant +\infty$$

for all E in Ω.

(*b*) We have $\mu(\varnothing) = \inf_{\cup C \supset \phi} \Sigma\tau(\mathscr{C}_i) \leqslant \Sigma\tau(\varnothing) = 0.$

Hence $\mu(\varnothing) = 0$.

(*c*) If $E_1 \subset E_2$, any cover of E_2 also covers E_1 and so

$$\mu(E_1) \leqslant \mu(E_2).$$

(*d*) Let $\{E_i\}$ be any sequence of sets of Ω. We prove that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leqslant \sum_{i=1}^{\infty} \mu(E_i). \tag{15}$$

This result is trivial if $\sum_{i=1}^{\infty} \mu(E_i) = +\infty.$

So we may suppose that $\sum_{i=1}^{\infty} \mu(E_i)$

is finite. Then, in particular, each $\mu(E_i)$ is finite. So, if $\epsilon > 0$ is given, for each integer $i > 1$ we can choose a sequence $\{C_j^{(i)}\}_{j=1}^{j=\infty}$ of sets in $\mathscr{C}$ with

$$E_i \subset \bigcup_{j=1}^{\infty} C_j^{(i)},$$

$$\sum_{j=1}^{\infty} \tau(C_j^{(i)}) \leqslant \mu(E_i) + \epsilon . 2^{-i}.$$

Let $\{D_i\}$ be a sequence obtained by rearranging the sets $C_j^{(i)}$ $(i, j = 1, 2, \ldots)$ as a single sequence. Then

$$\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} D_i,$$

$$D_i \in \mathscr{C} \quad (i = 1, 2, \ldots)$$