Which Banach spaces can be distinguished with the help of orthonormal systems?	Which orthonormal systems can be distinguished with the help of Banach spaces?

Introduction

The main goal of functional analysis is to provide powerful tools for a unified treatment of differential and integral equations, integral transforms, expansions and approximations of functions, and various other topics. A basic idea consists in extending classical results about real or complex functions to operators acting between topological linear spaces. Another important goal is the classification of objects, like spaces and operators. Luckily, these goals, the practical and the theoretical, are closely related to each other.

A significant trend in Banach space theory is the search for numerical parameters that can be used to quantify special properties. Certainly, everybody would agree that Hilbert spaces are the most beautiful among all Banach spaces. Thus it is important to decide whether a given Banach space admits an equivalent norm induced by an inner product. If so, this space is called *Hilbertian*. If such renormings do not exist, then we may ask for a measure of non-Hilbertness. To what extent is the sequence space l_4 closer to l_2 than l_{1892}?

We illustrate our point of view by asking whether Bessel's inequality also holds for functions $\boldsymbol{f}$ with values in a Banach space X. Do we have

$$\left(\sum_{k=1}^{n}\left\|\frac{1}{2\pi}\int_{-\pi}^{+\pi}\boldsymbol{f}(t)\exp(-ikt)\,dt\right\|^2\right)^{1/2} \leq \left(\frac{1}{2\pi}\int_{-\pi}^{+\pi}\|\boldsymbol{f}(t)\|^2\,dt\right)^{1/2} ? \qquad \text{(b)}$$

As observed by S. Bochner in 1933, this is not so in general. Later, it became clear that the validity of (b), even if it is only true for some fixed $n \geq 2$, characterizes Hilbert spaces isometrically; see [AMI, p. 51]. An isomorphic analogue of this criterion was established by S. Kwapień in 1972. He showed that X is Hilbertian if and only if there exists a constant $c \geq 1$ such that

$$\left(\sum_{k=1}^{n}\left\|\frac{1}{2\pi}\int_{-\pi}^{+\pi}\boldsymbol{f}(t)\exp(-ikt)\,dt\right\|^2\right)^{1/2} \leq c\left(\frac{1}{2\pi}\int_{-\pi}^{+\pi}\|\boldsymbol{f}(t)\|^2\,dt\right)^{1/2} \qquad \text{(B)}$$

for all square integrable X-valued functions $\boldsymbol{f}$ and $n = 1, 2, \ldots$. Now it is only a minor step to fix n and ask for the least constant $c \geq 1$ such that (B) holds in a given Banach space X. Denote, for the moment, this quantity by $\varphi_n(X)$. Then we have $\varphi_n(X) \leq \sqrt{n}$ and

$$\varphi_n(l_r) \asymp n^{|1/r-1/2|} \quad \text{for } 1 \leq r \leq \infty;$$

see p. 23 for the definition of $\asymp$. This observation suggests the following: With every exponent $0 \leq \lambda \leq 1/2$ we associate the class F_λ consisting of all Banach spaces X such that $\varphi_n(X) \prec n^\lambda$. Then $l_r \in \mathsf{F}_\lambda \setminus \mathsf{F}_{\lambda-\varepsilon}$ for $\lambda = |1/r - 1/2| \geq \varepsilon > 0$. Thus F_λ strictly increases with λ, and we have obtained a useful classification of Banach spaces. Remember that, by Kwapień's criterion, F_0 is the class of all Hilbertian Banach spaces.

Of course, we may wonder what happens when the trigonometric system is replaced by any other orthonormal system, complete or not. Important examples are Haar and Walsh functions, on the one hand, and Rademacher functions and Gaussian random variables, on the other hand.

We also mention that there are many different ways to obtain quantities similar to $\varphi_n(X)$. For instance, given two orthonormal systems $\mathcal{A}_n = (a_1, \dots, a_n)$ and $\mathcal{B}_n = (b_1, \dots, b_n)$ in Hilbert spaces $L_2(M, \mu)$ and $L_2(N, \nu)$, respectively, we can look for the least constant $c \geq 1$ such that

$$\left(\int_N \left\| \sum_{k=1}^n x_k b_k(t) \right\|^2 d\nu(t) \right)^{1/2} \leq c \left(\int_M \left\| \sum_{k=1}^n x_k a_k(s) \right\|^2 d\mu(s) \right)^{1/2},$$

where $x_1, \dots, x_n$ range over a Banach space X. An obvious modification allows us to extend this definition even to (bounded linear) operators acting from a Banach space X into a Banach space Y.

The asymptotic behaviour of the sequence $(\varphi_n(X))$ is invariant under isomorphisms. However, there are non-isomorphic Banach spaces X and Y such that $\varphi_n(X) \asymp \varphi_n(Y)$. For example, $\varphi_n(l_r \oplus l_2) \asymp \varphi_n(l_r)$. Thus we may ask which differences between X and Y are realized by φ_n. Roughly speaking, $\varphi_n(X)$ is determined by the *'worst'* n-dimensional subspace of X, where badness means large deviation from l_2^n. More generally, we may say that $\varphi_n(X)$ only depends on the collection of all n-dimensional subspaces, but neither on their position inside X nor on how often a specific subspace occurs.

In this book, we present a theory of orthonormal expansions with vector-valued coefficients and describe its interplay with Banach space geometry. Many results were obtained by straightforward extension of those concerned with Rademacher functions and Gaussian random variables. However, we hope that our general view yields more insight even into such well-known concepts as type and cotype of Banach spaces, B-convexity, superreflexivity, the vector-valued Fourier transform, the vector-valued Hilbert transform and the unconditionality property for martingale differences (UMD).

It is our hope that this treatise will be read not only by an esoteric group of specialists, but also by some graduate students interested in functional analysis. We have included many unsolved problems which show that there remains something to do for the future. Large parts of the presentation should be understandable with a basic knowledge in Banach space theory together with an elementary background in real analysis, probability and algebra. Exceptions prove the rule!

The proofs in this treatise require techniques from the fields just mentioned. Besides classical inequalities, we use various properties of special functions. Clearly, harmonic analysis serves as the basic pattern. It will turn out that orthonormal systems consisting of characters on compact Abelian groups possess many advantages because of the underlying algebraic structure. Another important feature is the use of probabilistic concepts, like random variables and martingales. In the theory of superreflexivity we employ Ramsey's theorem from combinatorics. Ultraproducts will prove to be an indispensable tool. Further key-words are: interpolation, extrapolation and averaging. Last but not least, we present many tricks and non-straightforward ideas. Of course, lengthy manipulations cannot be avoided. However, we have done our best to make things as easy as possible, and we hope the final result provides a colourful picture.

Basically, we have adopted standard notation and terminology from Banach space theory. It may nevertheless happen that experts well-acquainted with some special results are shocked by the symbols $\boldsymbol{\varrho}(T|\mathcal{B}_n, \mathcal{A}_n)$ and $\boldsymbol{\delta}(T|\mathcal{B}_n, \mathcal{A}_n)$ or even $\boldsymbol{\varrho}_u^{(v)}(T|\mathcal{B}_n, \mathcal{A}_n)$ and $\boldsymbol{\delta}_u^{(v)}(T|\mathcal{B}_n, \mathcal{A}_n)$. Hopefully, this displeasure will gradually be replaced by the understanding that our *lengthy notation* is indeed quite economical and suggestive. Of course, it seems better at first glance to denote the Rademacher cotype q constant computed with n vectors simply by $C_q(X, n)$ or $C_{q,n}(X)$, as done in [DIE*a, p. 290], [MIL*, p. 51] and [TOM, p. 188]. However, there occur similar quantities related to Gaussian random variables, various trigonometric functions, etc. Thus in the traditional way, we would run out of letters very quickly. To help the patient reader, a fairly complete list of symbols is included, pp. 514–522.

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Preliminaries

This chapter provides some elementary facts from the theory of Banach spaces and the basic terminology. For more information, we recommend the following books:

Beauzamy .... *Introduction to Banach spaces and their geometry* [BEA 2],
Day .... *Normed linear spaces* [DAY],
Dunford/Schwartz .... *Linear operators, vol. I* [DUN*1],
Lindenstrauss/Tzafriri... *Classical Banach spaces, vols. I and II* [LIN*1, LIN*2].

0.1 Banach spaces and operators

0.1.1 Throughout this book, X, Y and Z denote **Banach spaces** over $\mathbb{K}$ (synonym of the real field $\mathbb{R}$ or the complex field $\mathbb{C}$). Whenever it is necessary to indicate that x is an element of the Banach space X, then we denote its norm by $\|x|X\|$.

The **closed unit ball** of X is defined by $U_X := \{x \in X : \|x\| \le 1\}$.

CONVENTION. Unless otherwise stated, all Banach spaces under consideration are assumed to be different from $\{o\}$, where o denotes the zero element.

0.1.2 We write L for the **class of all Banach spaces**.

0.1.3 The **dual Banach space** X' consists of all (bounded linear) **functionals** $x' : X \to \mathbb{K}$. The value of x' at $x \in X$ is denoted by $\langle x, x' \rangle$, and we let

$$\|x'\| := \sup\{|\langle x, x'\rangle| : x \in U_X\}.$$

Moreover, U_X° stands for the closed unit ball of X'.

When dealing with duals of higher order, besides $\langle x, x' \rangle$ we use the symbols $\langle x'', x' \rangle$ and $\langle x'', x''' \rangle$. That is, $x \in X$ and $x'' \in X''$ are placed left, while $x' \in X'$ and $x''' \in X'''$ are placed right.

0.1.4 Throughout this book, T denotes a (bounded linear) **operator** from X into Y. The **null space** and the **range** of T are defined by

$$N(T) := \{ x \in X : Tx = o \} \quad \text{and} \quad M(T) := \{ Tx \in Y : x \in X \},$$

respectively. The **operator norm** is given by

$$\|T\| := \sup\{ \|Tx\| : x \in U_X \}.$$

Whenever it is advisable to indicate that the operator T acts from X into Y, then we use the more precise notation $\|T : X \to Y\|$. We denote the identity map of X by I_X. If $\|T\| \le 1$, then T is called a **contraction**.

The Banach space of all operators from X into Y is denoted by $\mathfrak{L}(X,Y)$. To simplify matters, we write $\mathfrak{L}(X)$ instead of $\mathfrak{L}(X,X)$.

0.1.5 Let $\mathfrak{L}$ denote the **class of all operators** acting between arbitrary Banach spaces. This means that

$$\mathfrak{L} = \bigcup_{X,Y} \mathfrak{L}(X,Y),$$

where X and Y range over L.

0.1.6 For $T \in \mathfrak{L}(X,Y)$, the **dual operator** $T' \in \mathfrak{L}(Y', X')$ is defined by

$$\langle x, T'y' \rangle = \langle Tx, y' \rangle \quad \text{for } x \in X \text{ and } y' \in Y'.$$

0.1.7 For fixed $x \in X$, the rule

$$K_X x : x' \longrightarrow \langle x, x' \rangle$$

defines a functional on X'. In this way, we obtain the **natural embedding** K_X from X into X'', which is a linear isometry. Note that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad T \quad} & Y \\ {\scriptstyle K_X}\big\downarrow & & \big\downarrow{\scriptstyle K_Y} \\ X'' & \xrightarrow[\quad T'' \quad]{} & Y'' \end{array}$$

commutes for every operator $T \in \mathfrak{L}(X,Y)$.

0.1.8 For a proof of the following classical result, we refer the reader to [DEF*, p. 73] and [PIE 2, p. 383].

HELLY'S LEMMA. *Let $x'' \in X''$. Then, given $x'_1, \dots, x'_n \in X'$ and $\varepsilon > 0$, there exists $x \in X$ such that*

$$\|x\| \leq (1+\varepsilon)\|x''\| \quad \textit{and} \quad \langle x, x'_k \rangle = \langle x'', x'_k \rangle \quad \textit{for } k = 1, \dots, n.$$

0.1.9 An operator $J \in \mathfrak{L}(X,Y)$ is an **injection** if there exists a constant $c > 0$ such that

$$\|Jx\| \geq c\,\|x\| \quad \text{for all } x \in X.$$

A **metric injection** is defined by the property that $\|Jx\| = \|x\|$.

An operator $Q \in \mathfrak{L}(X,Y)$ is a **surjection** if $Q(X) = Y$. By definition, a **metric surjection** $Q \in \mathfrak{L}(X,Y)$ maps the open unit ball of X onto the open unit ball of Y; see [PIE 2, pp. 26–28].

By a **subspace** M of a Banach space X we always mean a closed linear subset. The canonical (metric) injection from M into X is denoted by J_M^X. If N is a subspace of X, then Q_N^X stands for the canonical (metric) surjection from X onto the **quotient space** X/N.

An operator $P \in \mathfrak{L}(X)$ is called a **projection** if $P^2 = P$. Subspaces M of X that can be obtained as the range of a projection are said to be **complemented**.

0.1.10 A real or complex **Hilbert space** with the **inner product** $(\cdot,\cdot)$ will always be denoted by H or K.

0.1.11 With every element $y \in H$ we associate the functional

$$\overline{y} : x \rightarrow (x, y).$$

By the Riesz representation theorem, the map $C_H : y \rightarrow \overline{y}$ is a conjugate-linear isometry from H onto H'.

0.1.12 Let $T \in \mathfrak{L}(H,K)$, where H and K are Hilbert spaces. Then the **adjoint operator** $T^* \in \mathfrak{L}(K,H)$ is defined by

$$(x, T^*y) = (Tx, y) \quad \text{for } x \in H \text{ and } y \in K.$$

This means that $T^* = C_H^{-1} T' C_K$.

0.2 Finite dimensional spaces and operators

0.2.1 The dimension of a **finite dimensional** linear space M is denoted by $\dim(M)$. Given elements $x_1, \dots, x_n$ in any linear space X, then $\dim[x_1, \dots, x_n]$ stands for the dimension of $\operatorname{span}(x_1, \dots, x_n)$, the linear span.

A subspace N of a linear space X is said to be **finite codimensional** if $\operatorname{cod}(N) := \dim(X/N)$ is finite.

0.2.2 For a Banach space X the collection of all subspaces M with $\dim(M) \le n$ is denoted by $\mathrm{DIM}_{\le n}(X)$. Analogously, $\mathrm{COD}_{\le n}(X)$ stands for the collection of all subspaces N with $\operatorname{cod}(N) \le n$. We write

$$\mathrm{DIM}(X) := \bigcup_{n=0}^{\infty} \mathrm{DIM}_{\le n}(X) \quad \text{and} \quad \mathrm{COD}(X) := \bigcup_{n=0}^{\infty} \mathrm{COD}_{\le n}(X).$$

0.2.3 The **Banach–Mazur distance** of n-dimensional Banach spaces X and Y is defined by

$$d(X,Y) := \inf\Big\{ \|T\|\,\|T^{-1}\| \,:\, T \in \mathfrak{L}(X,Y), \text{ bijection}\Big\}.$$

We have a multiplicative triangle inequality $d(X,Z) \le d(X,Y)\,d(Y,Z)$. Moreover, X and Y are isometric if and only if $d(X,Y) = 1$.

Whenever there exist $T \in \mathfrak{L}(X,Y)$ and $0 < c < 1$ such that

$$\Big|\|Tx\| - \|x\|\Big| \le c\|x\| \quad \text{for} \quad x \in X,$$

then $\|Tx\| \le (1+c)\|x\|$ and $(1-c)\|x\| \le \|Tx\|$. Hence $d(X,Y) \le \frac{1+c}{1-c}$.

0.2.4 Without proof, we state an extremely important result; see [joh], [PIE 2, p. 385] and [TOM, p. 54]. As usual, l_2^n denotes the n-dimensional Hilbert space; see 0.3.2.

JOHN'S THEOREM. *$d(X, l_2^n) \le \sqrt{n}$ whenever* $\dim(X) = n$.

0.2.5 An operator $T \in \mathfrak{L}(X,Y)$ has **finite rank** if its range

$$M(T) := \{Tx : x \in X\}$$

is finite dimensional. Then we write $\operatorname{rank}(T) = \dim(M(T))$. The set of all finite rank operators from X into Y is denoted by $\mathfrak{F}(X,Y)$.

0.3 Classical sequence spaces

0.3.1 Given any set $\mathbb{I}$, by an **$\mathbb{I}$-tuple** we mean a family of objects indexed by $i \in \mathbb{I}$. The letter $\mathbb{F}$ always stands for a finite index set, and $|\mathbb{F}|$ denotes its cardinality.

0.3.2 Let $1 \le r < \infty$, and consider any $\mathbb{I}$-tuple of Banach spaces X_i. Then $[l_r(\mathbb{I}), X_i]$ consists of all $\mathbb{I}$-tuples (x_i) with $x_i \in X_i$ for which

$$\|(x_i)|l_r(\mathbb{I})\| := \Big(\sum_{\mathbb{I}} \|x_i\|^r\Big)^{1/r}$$

is finite. In the limiting case $r = \infty$, the $\mathbb{I}$-tuples (x_i) are assumed to be bounded, and we let

$$\|(x_i)|l_\infty(\mathbb{I})\| := \sup_{\mathbb{I}} \|x_i\|.$$

To simplify matters, we write $[l_r, X_i]$ and $[l_r^n, X_i]$ when the index set $\mathbb{I}$ is $\{1, 2, \ldots\}$ and $\{1, \ldots, n\}$, respectively. In the scalar-valued case, the usual symbols $l_r(\mathbb{I})$, l_r and l_r^n will be used. The Banach space $[l_r(\mathbb{I}), X_i]$ is called the **$l_r(\mathbb{I})$-sum** of (X_i). If $X_i = X$ for all $i \in \mathbb{I}$, we refer to $[l_r(\mathbb{I}), X]$ as the **$l_r(\mathbb{I})$-multiple** of X. In this case, the underlying norm will sometimes be denoted by the more precise symbol $\|(x_i)|[l_r(\mathbb{I}), X]\|$.

0.3.3 The **natural injection** J_k^X from X_k into $X := [l_r(\mathbb{I}), X_i]$ takes $x \in X_k$ into the $\mathbb{I}$-tuple (x_i) with $x_k = x$ and $x_i = o$ for $i \ne k$. The **natural surjection** Q_k^X from X onto X_k is defined by $Q_k^X(x_i) := x_k$.

0.3.4 Let (T_i) be any $\mathbb{I}$-tuple of operators $T_i \in \mathfrak{L}(X_i, Y_i)$. Then the rule

$$[l_r(\mathbb{I}), T_i] : (x_i) \longrightarrow (T_i x_i)$$

yields a **diagonal operator** from $[l_r(\mathbb{I}), X_i]$ into $[l_r(\mathbb{I}), Y_i]$ provided that

$$\big\|[l_r(\mathbb{I}), T_i]\big\| = \sup_{\mathbb{I}} \|T_i\|$$

is finite. With the natural injections and surjections introduced above, we have

$$T_j = Q_j^Y [l_r(\mathbb{I}), T_i] J_j^X \quad \text{whenever } j \in \mathbb{I}.$$

The operators

$$\begin{aligned} B_\alpha &: (x_k) \longrightarrow ((1 + \log k)^{-\alpha} x_k), \\ C_\alpha &: (x_k) \longrightarrow (k^{-\alpha} x_k), \\ D_\alpha &: (x_k) \longrightarrow (2^{-k\alpha} x_k), \end{aligned}$$

defined for $(x_k) \in [l_2, l_\infty^{2^k}]$ and $\alpha \ge 0$ will play an important role as examples; see 1.2.12.

0.4 Classical function spaces

0.4.1 Throughout this book, (M,μ) and (N,ν) are assumed to be σ-finite measure spaces. To simplify notation, we suppress the underlying σ-algebras of measurable subsets. Scalar-valued functions will be denoted by $f, g, \ldots$, while $\boldsymbol{f}, \boldsymbol{g}, \ldots$ stand for vector-valued functions. With every scalar-valued function f on $M \times N$ we associate the function $\boldsymbol{f}$ which assigns to any fixed $s \in M$ the function $t \to f(s,t)$.

As usual, Banach function spaces are constituted by equivalence classes of functions which coincide almost everywhere.

0.4.2 If $f_1, \ldots, f_n$ are scalar-valued functions on M and $x_1, \ldots, x_n \in X$, then we let

$$\sum_{k=1}^{n} f_k \otimes x_k : s \longrightarrow \sum_{k=1}^{n} f_k(s)\, x_k.$$

0.4.3 A function $\boldsymbol{f} : M \to X$ is called **simple** if it can be written in the form

$$\boldsymbol{f} = \sum_{k=1}^{n} \chi_k \otimes x_k,$$

where $\chi_1, \ldots, \chi_n$ are characteristic functions of measurable subsets $A_1, \ldots, A_n$. We denote by $S_0(M,\mu) \otimes X$ the collection of such simple functions for which $\mu(A_1), \ldots, \mu(A_n)$ are finite.

0.4.4 A function $\boldsymbol{f} : M \to X$ that coincides almost everywhere with the pointwise limit of a sequence of simple X-valued functions is said to be **measurable**.

0.4.5 For $1 \le r < \infty$, we denote by $[L_r(M,\mu), X]$ the Banach space of all measurable functions $\boldsymbol{f} : M \to X$ for which

$$\|\boldsymbol{f}|L_r\| := \left(\int_M \|\boldsymbol{f}(s)\|^r \, d\mu(s) \right)^{1/r}$$

is finite. In the limiting case $r = \infty$, we let $[L_\infty(M,\mu), X]$ denote the Banach space of all essentially bounded measurable functions $\boldsymbol{f} : M \to X$ equipped with the norm

$$\|\boldsymbol{f}|L_\infty\| := \operatorname*{ess-sup}_M \|\boldsymbol{f}(s)\|.$$

When X is the scalar field $\mathbb{K}$, we simply write $L_r(M,\mu)$ instead of $[L_r(M,\mu), \mathbb{K}]$.

We denote by $L_r(M,\mu)\otimes X$ the collection of all X-valued functions

$$\sum_{k=1}^{n} f_k \otimes x_k$$

with $f_1,\dots,f_n \in L_r(M,\mu)$ and $x_1,\dots,x_n \in X$. Note that $L_r(M,\mu)\otimes X$ is a dense linear subset of $[L_r(M,\mu),X]$.

REMARK. For further information about Banach spaces $[L_r(M,\mu),X]$, the reader is referred to [DUN*1, pp. 119 and 146].

CONVENTIONS. From a pedantic point of view, the norm on $[L_r,X]$ should be denoted by $\|\cdot|[L_r,X]\|$. But, for simplicity, we will mostly use the shorter symbol $\|\cdot|L_r\|$. However, if a vector-valued function $\boldsymbol{f}\in[L_u(M,\mu),L_v(N,\nu)]$ with $1\le u,v<\infty$ is regarded as a scalar-valued function f on $M\times N$, then we will write

$$\|\boldsymbol{f}|[L_u,L_v]\| := \left[\int_M \Big(\int_N |f(s,t)|^v\, d\nu(t)\Big)^{u/v} d\mu(s)\right]^{1/u}.$$

When Banach function spaces $L_r(M,\mu)$ are used as examples, we simply write L_r and assume tacitly that they are infinite dimensional.

0.4.6 For $1<r<\infty$, the **dual exponent** r' is defined by $\frac{1}{r}+\frac{1}{r'}=1$. In the cases $r=1$ and $r=\infty$, we let $r'=\infty$ and $r'=1$, respectively.

The famous **Hölder inequality** says that

$$\left|\int_M f(s)g(s)\,d\mu(s)\right| \le \Big(\int_M |f(s)|^r\,d\mu(s)\Big)^{1/r}\Big(\int_M |g(s)|^{r'}\,d\mu(s)\Big)^{1/r'}$$

for $f\in L_r(M,\mu)$ and $g\in L_{r'}(M,\mu)$. If g is not the zero function, then equality holds if and only if there exist constants $c\ge 0$ and $\tau\in\mathbb{R}$ such that

$$|f(s)|^r = c\,|g(s)|^{r'} \quad\text{and}\quad f(s)g(s)e^{i\tau}\ge 0$$

for almost all $s\in M$ and $g(s)\ne 0$; see [ZYG, vol. I, p. 18].

0.4.7 Given $\boldsymbol{f}\in[L_r(M,\mu),X]$ and $\boldsymbol{g}\in[L_{r'}(M,\mu),X']$, we write

$$\langle \boldsymbol{f},\boldsymbol{g}\rangle := \int_M \Big\langle \boldsymbol{f}(s),\boldsymbol{g}(s)\Big\rangle\, d\mu(s).$$

If $\boldsymbol{g}\in[L_{r'}(M,\mu),X']$ is fixed, then the rule

$$\boldsymbol{f} \longrightarrow \langle \boldsymbol{f},\boldsymbol{g}\rangle$$