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1 Introduction

EARLY LIFE

Frege was born in 1848 in Wismar, a small port on the Baltic coast in Mecklenberg.[1] His father, who ran a private school for girls there, died when he was eighteen, and his mother took over the running of the school in order to be able to provide for the university education of Frege and his younger brother. Frege was encouraged in this by a young teacher at his father's school called Leo Sachse. Sachse had attended university in Jena, and Frege went there too in 1869, lodging in the same room that Sachse had rented there before him. Frege's studies in Jena consisted mainly of courses in mathematics and chemistry. The only philosophy was a course on Kant's critical philosophy given by Kuno Fischer.

From Jena Frege went on to Göttingen, where he took further courses in mathematics and physics and wrote a dissertation, 'On a Geometrical Representation of Imaginary Forms in the Plane'. His only philosophy course at Göttingen was one on the philosophy of religion given by Hermann Lotze. After five semesters, Frege returned to Jena to submit a further dissertation for his venia docendi (i.e. licence to teach in the university). The title of this second dissertation was 'Methods of Calculation based on an Extension of the Concept of Quantity'. Neither dissertation exhibits more than a passing interest in logic or the philosophy of mathematics.

One of Frege's mathematics lecturers at Jena, Ernst Abbe, acted as a sort of mentor, supporting him, for instance, in his efforts to

1 For information about Frege's life I have relied throughout this Introduction on Lothar Kreiser, *Frege: Leben, Werk, Zeit* (Hamburg: Meiner, 2001).

gain promotion. But it is hard to find anyone in Frege's education who might count as a philosophical teacher of central importance. The nearest to a direct influence is perhaps Lotze, not because of his lectures on the philosophy of religion but because he published a book on logic in 1874. Dummett has convincingly argued[2] that an undated list of seventeen numbered observations about logic which has survived in Frege's hand was written in response to reading Lotze's book; internal evidence strongly suggests that these notes are probably among the earliest of Frege's unpublished writings on logic to have survived (although perhaps not quite pre-dating *Begriffsschrift*, as Dummett suggested).[3]

In the notes, Frege makes a distinction, which was to be central to his thinking about logic throughout his career, between thoughts and ideas: a thought is something such that 'it makes sense to ask whether it is true or untrue', whereas 'associations of ideas are neither true nor untrue'. Truth is objective. As Frege puts it, '2 times 2 is 4' is true, and will continue to be so even if, as a result of Darwinian evolution, human beings were to come to assert that 2 times 2 is 5. Every truth is eternal and independent of being thought by anyone and of the psychological make-up of anyone thinking it.[4]

Frege does not yet quite say, as he would later, that the subject-matter of logic is truth, but he does say that logic 'only becomes possible with the conviction that there is a difference between truth and untruth'. Following close on this, given that truth is objective, is that logic is not a branch of psychology. 'No psychological investigation can justify the laws of logic.' But truth, which is on Frege's presentation fundamental to logic, cannot be defined. 'What true is,' he says, 'is indefinable.' Frege does not at this stage give an argument to explain *why* truth is indefinable, but he later held that any attempt to define it would inevitably be circular, because one would have to understand the definition as being *true*.

If what I have said about the dating of these notes is correct, then Frege formed some of his fundamental views about logic remarkably early. It is worth stressing, moreover, that the views just mentioned

[2] M. Dummett, 'Frege's Kernsätze zur Logik', in his *Frege and Other Philosophers* (Oxford: Clarendon Press, 1991).
[3] See Frans Hovens, 'Lotze and Frege: The dating of the "Kernsätze"', *History and Philosophy of Logic*, 18 (1997), pp. 17–31.
[4] *PW*, p. 174.

constitute a response to Lotze's book, not a summary of it. It is true, for instance, that Lotze distinguished between logic and psychology, but his reason for doing so was that logic deals with the value of our thoughts whereas psychology deals with their genesis. This is obviously rather distant from Frege's anti-psychologism, which was based on the objectivity of truth, not on its value.[5]

BEGRIFFSSCHRIFT

Frege's short book *Begriffsschrift*, which he published in 1879, marks the beginning of modern logic. The word 'Begriffsschrift' is not Frege's own, but seems to have been coined by Humboldt in 1824:[6] it is usually translated 'conceptual notation' or 'concept-script'. Here we shall call the book by its italicized German title and use the word unitalicized for the formal language it describes. The idea of a formal language is not itself new with Frege. But Frege's Begriffsschrift has a number of features that were quite new in 1879.

The 'seventeen key sentences' already show Frege treating logic as a subject whose central concern is truth, and regarding thoughts as of relevance to logic because they are what truth applies to. In the first chapter of *Begriffsschrift* ('Definition of the symbols'), Frege uses the term 'judgeable content' for what he previously called a thought. Moreover, he straightaway highlights an issue which was to remain of concern to him throughout his philosophical writings, namely that of identifying the structure of a judgeable content. Since what follows logically from

The Greeks defeated the Persians at Plataea

and what follows from

The Persians were defeated by the Greeks at Plataea

are identical, logic need not distinguish between these two propositions: they have the same judgeable content.

[5] For the view that Frege should be seen as a neo-Kantian who was heavily influenced by Lotze, see G. Gabriel, 'Frege als Neukantianer', *Kant-Studien*, 77 (1986), pp. 84–101. See also Hans Sluga, *Gottlob Frege* (London: Routledge and Kegan Paul, 1980).

[6] See M. Beaney and Erich H. Reck (eds.), *Gottlob Frege: Critical Assessments of Leading Philosophers* (London: Routledge, 2005), vol II, p. 13.

One of Frege's innovations was to introduce a sign to mark the act of judging that something is the case. The sign he used was a vertical line which he called the judgement stroke. He also made use of a horizontal line which he called the content stroke, whose purpose was to turn what follows the stroke into a judgeable content. However, it is not entirely clear what this amounts to. A charitable reader[7] might see this as an implicit recognition that anything which expresses a judgeable content is of necessity complex, and hence in need of binding into a unity before it is capable of being judged. This is at any rate something which Frege was in his later writings keen to assert. A less charitable reader might think that if I have expressed a content then that is all there is to it: if the content I have expressed is judgeable, nothing more is needed to indicate that; if it is not, then preceding it with a stroke cannot make it so.

Because in practice the vertical judgement stroke never occurs without being immediately followed by the horizontal content stroke, the combination of the two strokes inevitably came to be treated as a symbol in its own right. This is the origin of the turnstile symbol $\vdash$ that is ubiquitous in modern logic. However, it is worth stressing that this symbol, although it originated with Frege, is often now used in ways that he would not have recognized. In particular, Frege did not recognize a notion of conditional assertion, so would not have allowed the turnstile to be embedded, as in expressions such as

$$A_1, A_2, \ldots, A_n \vdash B.$$

The second major innovation which Frege's conceptual notation encapsulates – and the one for which it is nowadays renowned – is a method for expressing multiple generality. However, Frege not only provides such a notation; he also displays a firm grasp of the principles that underlie it. He is clear, for instance, that in a quantified expression such as $\forall x \exists y Rxy$ the letters 'x' and 'y' do not function like names. Frege conspicuously avoids the unfortunate usage inherited from mathematics which refers to them as variables: as he makes clear, they are not variable names but placeholders.

[7] E.g. Peter Sullivan, 'Frege's logic', in Dov M. Gabbay and John Woods (eds.), *Handbook of the History of Logic*, vol. III (Amsterdam: North-Holland, 2004), pp. 659–750.

If, in an expression (whose content need not be assertible), a simple or complex symbol occurs in one or more places and we imagine it as replaceable by another ... then we call the part of the expression that shows itself invariant a function and the replaceable part its argument.[8]

Notice, incidentally, that on this account predicates are a particular kind of function, namely those derived from expressions whose content is assertible (i.e. from sentences).

Frege's choice of symbols shows awareness, too, of the desirability of notational economy. He has a sign for the universal quantifier (nowadays always notated $\forall$), but he does not also have a sign for the existential quantifier $\exists$, since $\exists x$ can easily be regarded as an abbreviation for $\sim\forall x\sim$. The same economy is evident too in his choice of propositional connectives. He has signs for negation (nowadays $\sim$) and for material implication (nowadays $\rightarrow$) but not for the other connectives, which can be defined in terms of them. He also notes explicitly that he could just as well have used negation and conjunction, although he stops just short of asserting that they are adequate to express all the others. Although he did not actually make use of the device of truth-tables in presenting his account, he might as well have done, as his presentation of the meanings of the logical connectives is explicitly truth-functional in character.

The other thing for which the *Begriffsschrift* is especially notable is the axiom system for predicate calculus contained in the second chapter ('Representation and derivation of some judgements of pure thought'). He had already in the first chapter formulated *modus ponens*

From $\vdash B \rightarrow A$ and $\vdash B$ derive $\vdash A$

as well as the quantifier rule

From $\vdash \forall x(A \rightarrow \Phi(x))$ derive $\vdash A \rightarrow \forall x \Phi(x)$.

Now he added the logical axioms, which he arranges in four groups:

$$\vdash a \rightarrow (b \rightarrow a)$$
$$\vdash (c \rightarrow (b \rightarrow a)) \rightarrow ((c \rightarrow b) \rightarrow (c \rightarrow a))$$
$$\vdash (d \rightarrow (b \rightarrow a)) \rightarrow (b \rightarrow (d \rightarrow a))$$

[8] *Bs*, §9.

$$\vdash (b \to a) \supset (\sim a \to \sim b)$$
$$\vdash \sim\sim a \to a$$
$$\vdash a \to \sim\sim a$$

$$\vdash c = d \to f(c) = f(d)$$
$$\vdash c = c$$

$$\vdash \forall x f(x) \to f(c)$$

Frege's presentation of his axiom system is curiously understated, however. The axioms appear in the text as numbered formulae, not distinguished by any mark from the other formulae which he states as being derivable from them. He identifies his preferred axiom system only indirectly, by listing the numbers of the formulae which form what he calls the 'core' of his system.

The third chapter of *Begriffsschrift,* 'Some topics from a general theory of sequence', is a treatment of the theory of ancestral relations, expressed in the Begriffsschrift. Frege intended this chapter as an illustration of the power and elegance of his notation. There is no denying, however, that there is something unsatisfactory about the presentation. It offers its treatment of mathematical induction as an example of the ability of the Begriffsschrift to capture mathematical concepts and arguments, but then the chapter ends abruptly and in an oddly inconclusive manner. And the principle of mathematical induction itself is offered in a curiously understated way: it is not labelled as such, and the only indication in the text that this is what it is is Frege's observation that the Sorites paradox may be derived using it; mathematical induction is mentioned by name only in a laconic footnote.

RECEPTION

For all its many remarkable features, *Begriffsschrift* is undoubtedly a flawed work. One weakness, already noted, is its lack of clarity about the axiomatization of logic that it contains. Another is the rather lame presentation of the third chapter. But the feature that was of overriding importance in determining how the book would be received is one that we have not yet mentioned. In the exposition in the last section I used the symbols ~, →, ∀ for the logical

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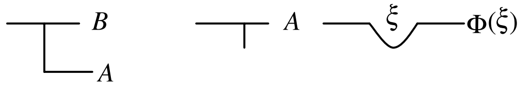
constants that are now common among logicians. But Frege did not use these symbols. What we would write as $A \rightarrow B$, $\sim A$ and $\forall x\ \Phi(x)$ he wrote as



respectively. Now Frege's two-dimensional notation no doubt has its advantages. Once the eye has become used to it, it exhibits the logical structure of a complicated expression more vividly than does the bracketing of the conventional, one-dimensional alternative. But the plain fact is that it was too radical a departure from what was familiar to have any hope of adoption, and no one other than Frege ever used it. Moreover, he himself was curiously stubborn about it. A more concessive personality than his might have responded to criticism by separating out the part that is most unfamiliar (the two-dimensionality) and asking his readers to focus on his other innovations, which are independent of it.

Perhaps financial pressure contributed to Frege's decision to publish the *Begriffsschrift* when he did, despite its evident incompleteness. Not only was the University of Jena in a poor financial state, but his own position within that institution was by no means secure. He was surviving as a *Privatdozent*, financially dependent on the fees paid by his students. Since the courses Frege gave were not popular, his income was small and highly variable from semester to semester. During the academic year 1878–9, for instance, it amounted to 249 marks.

The publication of *Begriffsschrift* seems to have had the desired effect of helping Frege's career. At any rate, in 1881 the university granted him an annual stipend of 300 marks. At this time his mother moved from Wismar to Jena, and they shared a house together for some years, which may also have aided his financial position.

The preface to *Begriffsschrift* promises that a work which applies the Begriffsschrift to arithmetic is imminent. And in the summer of 1882 Frege wrote to Stumpf, a contemporary of his then working at Prague:[9] 'I have now nearly completed a book in which

[9] The letter is presented in *PMC* as being to Anton Marty, but the editors acknowledge that the addressee may well have been Stumpf, since the letter from him quoted below is evidently a reply to it.

I treat the concept of number and demonstrate that the first principles of computation which up to now have generally been regarded as unprovable axioms can be proved from definitions by means of logical laws alone.'

Frege's confidence that he could indeed derive the truths of arithmetic 'from definitions by means of logical laws alone' arose from the application of his Begriffsschrift, which, he said, 'will not let through anything that was not expressly presupposed, even if it seems so obvious that in ordinary thought we do not even notice that we are relying on it for support'.

Frege's letter also shows the first signs of what was to be a continuing theme in his life, namely his feeling that his work was not receiving the attention from others that was its due. Stumpf's reply asked Frege, presumably in response to this complaint, 'whether it would not be appropriate to explain your line of thought first in ordinary language and then – perhaps separately on another occasion … – in the Begriffsschrift: I should think that this would make for a more favourable reception of both accounts.'

GRUNDLAGEN

Frege took Stumpf's advice. His attempt to 'explain [his] line of thought in ordinary language' resulted in what many consider to be his masterpiece, *Die Grundlagen der Arithmetik* (*The Foundations of Arithmetic*). In the period when he was writing the *Grundlagen*, between 1882 and 1884, Frege was teaching part-time at the Pfeiffer Institute, a private school in Jena, and indeed he mentions a book by Grassmann that was intended for use in schools.

In the *Grundlagen* Frege criticizes various views that had been offered on the nature of numbers and of arithmetical truths, before sketching his own account. Chief among the views Frege criticizes is Kant's, that the truths of arithmetic are synthetic a priori. Frege's principal objection to Kant's view is that it does not explain the scope of arithmetic. If arithmetic were synthetic, it would depend on intuition, and all our intuitions, according to Kant, are ultimately dependent on the structure of space and time. So arithmetic, since derived from the spatio-temporal structure of reality, would be applicable only to it. Yet, Frege says, the scope of arithmetic is wider. In this respect Frege distinguished arithmetic from geometry.

> The truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy. The wildest visions of delirium, the boldest inventions of legend and poetry, where animals speak and stars stand still, where men are turned to stone and tress turn into men, where the drowning haul themselves up out of swamps by their own topknots – all these remain, so long as they remain intuitable, still subject to the axioms of geometry.[10]

In this respect, Frege believed, geometry differs from arithmetic. Here, he said,

> we have only to try denying any one of our assumptions, and complete confusion ensues. Even to think at all seems no longer possible … The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought?[11]

Frege invites us to think, then, that, if arithmetic has the same range of applicability as logic itself (namely, everything thinkable), the explanation for this can only be that arithmetic is derivable from logic.

So far, though, what Frege had done was only to render this central claim plausible, not to prove it. Frege now turned to his attempt at a positive account of arithmetic as derived from logic. The first thing he did was to make the important observation that ascriptions of number do not apply to piles of stuff in the world. Before we can count, we need to know what it is we are counting. We need, that is to say, a concept. It is not the pack of cards itself that has the number fifty-two but the concept 'card in the pack'. The concept 'suit in the pack', by contrast, has the number four. This observation is no doubt obvious as soon as it is made, but to realize its importance one has only to read the confused writings of authors who failed to make it. Frege himself was probably helped to realize the point by a now-forgotten philosopher called Herbart (referred to by Frege briefly in a footnote), who said something similar, although rather less clearly, in 1825.[12]

Ascriptions of number are therefore on Frege's account second-level concepts. The first-level concept 'card in the pack' falls under the second-level concept 'having fifty-two instances'. More

[10] *Gl*, §14.
[11] *Ibid.*
[12] See D. Sullivan, 'Frege on the statement of number', *Philosophy and Phenomenological Research*, 50 (1990), pp. 595–603.

generally, the second-level concept 'having n instances' is called a *numerically definite quantifier*: 'F has n instances' is abbreviated to $\exists^n xFx$. Frege now considers what at first sight looks like the promising proposal that we should define numbers implicitly by means of the numerically definite quantifiers, which can be defined recursively as follows:

$$\exists^0 xFx =_{\mathrm{Df}} {\sim}\exists xFx;$$

$$\exists^{n+1} xFx =_{\mathrm{Df}} \exists x(Fx \,\&\, \exists^n y(Fy \,\&\, y \neq x)).$$

From these definitions it is possible to prove various arithmetical laws using logic alone, and hence, it seems, to vindicate the logicist thesis. Indeed, Frege's presentation encourages the thought that he intends a treatment of arithmetic on something like these lines, since in the Introduction to the book he lays some stress on the injunction 'never to ask for the meaning of a word in isolation, but only in the context of a proposition'. This injunction, which is nowadays known as the Context Principle, seems on the face of it to be designed precisely to license implicit definitions such as the one just offered: the definition does not tell us explicitly what the numbers are, but allows us to eliminate numerals progressively from contexts in which they occur.

However, Frege's guiding principle in his search for an account of arithmetic was that numbers are self-subsistent objects. This principle places a constraint on the use of the Context Principle, since Frege took it as central to objecthood that there should be a principle of individuation that enables us to recognize the same object again. If we are to introduce a term to refer to an object, therefore, we must give its identity conditions: our definition must suffice to determine whether the object introduced is the same as or different from any other object already known to us.

And this creates a problem for the account in terms of numerically definite quantifiers, since the implicit definition does not suffice to determine whether the numbers are equal to other objects. For instance, it does not, to use Frege's 'crude example', determine whether Julius Caesar is a natural number. Hence, Frege thought, the account must be rejected.

Frege now turned instead to the consideration of another proposal, namely that we should derive the basic laws of arithmetic from what is often now called Hume's Principle, i.e. the principle