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Volume 91

# Geometry of Sporadic Groups II
## Representations and amalgams

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

# *Geometry of Sporadic Groups II*
# *Representations and Amalgams*

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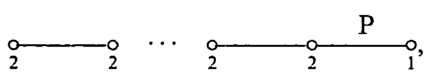
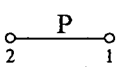
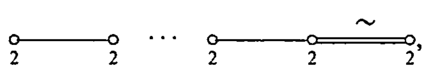
# Contents

# Preface

This is the second volume of the two-volume series which contains the proof of the classification of the flag-transitive $P$- and $T$-geometries. A $P$-geometry (Petersen geometry) has diagram



where  denotes the geometry of 15 edges and 10 vertices of the Petersen graph. A $T$-geometry (Tilde geometry) has diagram



where  denotes the 3-fold cover of the generalized quadrangle of order $(2,2)$, associated with the non-split extension $3 \cdot S_4(2) \cong 3 \cdot Sym_6$.

The final result of the classification, as announced in [ISh94b], is the following (we write $\mathscr{G}(G)$ for the $P$- or $T$-geometry admitting $G$ as a flag-transitive automorphism group).

**Theorem 1** *Let $\mathscr{G}$ be a flag-transitive $P$- or $T$-geometry and $G$ be a flag-transitive automorphism group of $\mathscr{G}$. Then $\mathscr{G}$ is isomorphic to a geometry $\mathscr{H}$ in Table I or II and $G$ is isomorphic to a group $H$ in the row corresponding to $\mathscr{H}$.*

In the first volume [Iv99] and in [IMe99] for the case $\mathscr{G}(J_4)$ the following has been established (for the difference between coverings and 2-coverings cf. Section 1.2).

**Theorem 2** *Let $\mathscr{H}$ be a geometry from Table I or II of rank at least 3 and $H$ be a group in the row corresponding to $\mathscr{H}$. Then*

(i) *$\mathcal{H}$ exists and is of correct type (i.e., P- or T-geometry);*
(ii) *$H$ is a flag-transitive automorphism group of $\mathcal{H}$;*
(iii) *suppose that $\widetilde{\mathcal{H}}$ is a P- or T-geometry, $\widetilde{H}$ is a flag-transitive automorphism group of $\widetilde{\mathcal{H}}$, $\varphi : \widetilde{\mathcal{H}} \to \mathcal{H}$ is a 2-covering which commutes with the action of $\widetilde{H}$ and the induced action of $\widetilde{H}$ on $\mathcal{H}$ coincides with $H$, then either $\varphi$ is an isomorphism or one of the following holds:*
  (a) *$\widetilde{\mathcal{H}} \cong \mathcal{G}(3 \cdot M_{22})$, $\mathcal{H} \cong \mathcal{G}(M_{22})$, $\widetilde{H} \cong 3 \cdot M_{22}$ or $3 \cdot \mathrm{Aut}\, M_{22}$ and $\varphi$ is a covering;*
  (b) *$\widetilde{\mathcal{H}} \cong \mathcal{G}(3^{23} \cdot Co_2)$, $\mathcal{H} \cong \mathcal{G}(Co_2)$, $\widetilde{H} \cong 3^{23} \cdot Co_2$ and $\varphi$ is not a covering;*
  (c) *$\widetilde{\mathcal{H}} \cong \mathcal{G}(3^{4371} \cdot BM)$, $\mathcal{H} \cong \mathcal{G}(BM)$, $\widetilde{H} \cong 3^{4371} \cdot BM$ and $\varphi$ is not a covering, in particular,*
(iv) *either $\mathcal{H}$ is simply connected or $\mathcal{H} \cong \mathcal{G}(M_{22})$ and the universal cover of $\mathcal{H}$ is $\mathcal{G}(3 \cdot M_{22})$.*

**Table I. Flag-transitive *P*-geometries**

| Rank | Geometry $\mathcal{H}$ | Flag-transitive automorphism groups $H$ |
|---|---|---|
| 2 | $\mathcal{G}(Alt_5)$ | $Alt_5$, $Sym_5$ |
| 3 | $\mathcal{G}(M_{22})$ | $M_{22}$, $\mathrm{Aut}\, M_{22}$ |
| | $\mathcal{G}(3 \cdot M_{22})$ | $3 \cdot M_{22}$, $3 \cdot \mathrm{Aut}\, M_{22}$ |
| 4 | $\mathcal{G}(M_{23})$ | $M_{23}$ |
| | $\mathcal{G}(Co_2)$ | $Co_2$ |
| | $\mathcal{G}(3^{23} \cdot Co_2)$ | $3^{23} \cdot Co_2$ |
| | $\mathcal{G}(J_4)$ | $J_4$ |
| 5 | $\mathcal{G}(BM)$ | $BM$ |
| | $\mathcal{G}(3^{4371} \cdot BM)$ | $3^{4371} \cdot BM$ |

If $\mathcal{F}$ is a geometry and $F$ is a flag-transitive automorphism group of $\mathcal{F}$ then $\mathcal{A}(F, \mathcal{F})$ denotes the amalgam of maximal parabolics associated with the action of $F$ on $\mathscr{F}$. In these terms the main result of this second volume can be stated as follows:

**Theorem 3** *Let $\mathscr{G}$ be a flag-transitive $P$- or $T$-geometry of rank at least 3 and G be a flag-transitive automorphism group of $\mathscr{G}$. Then for a geometry $\mathscr{H}$ and its automorphism group $H$ from Table I or II we have the following:*

$$\mathscr{A}(G,\mathscr{G}) \cong \mathscr{A}(H,\mathscr{H}).$$

In the above theorem we can assume that $\mathscr{H}$ is simply connected. Then by Theorem 1.4.5, $\mathscr{H}$ is the universal cover of $\mathscr{G}$ and $H$ is the universal completion of $\mathscr{A}(G,\mathscr{G})$.

Notice that Theorem 3 immediately implies the following

**Corollary 4** *Let $\mathscr{H}$ be a geometry from Table I or II and let $H$ be a flag-transitive automorphism group of $\mathscr{H}$. Then $H$ is one of the groups in the row corresponding to $\mathscr{H}$ and either*

(i) *$H$ is the full automorphism group of $\mathscr{H}$, or*
(ii) *$\mathscr{H} \cong \mathscr{G}(M_{22})$ or $\mathscr{G}(3 \cdot M_{22})$ and $H \cong M_{22}$ or $3 \cdot M_{22}$, respectively (so that $H$ is the unique self-centralized subgroup of index 2 in the automorphism group of $\mathscr{H}$).*

**Table II. Flag-transitive $T$-geometries**

| Rank | Geometry $\mathscr{H}$ | Flag-transitive automorphism groups $H$ |
|---|---|---|
| 2 | $\mathscr{G}(3 \cdot S_4(2))$ | $3 \cdot Alt_6$, $3 \cdot S_4(2) \cong 3 \cdot Sym_6$ |
| 3 | $\mathscr{G}(M_{24})$<br>$\mathscr{G}(He)$ | $M_{24}$<br>$He$ |
| 4 | $\mathscr{G}(Co_1)$ | $Co_1$ |
| 5 | $\mathscr{G}(M)$ | $M$ |
| $n$ | $\mathscr{G}(3^{\left[{n \atop 2}\right]_2} \cdot S_{2n}(2))$ | $3^{\left[{n \atop 2}\right]_2} \cdot S_{2n}(2)$ |

Now in order to deduce Theorem 1 from Theorems 2 and 3 it is sufficient to prove the following

**Proposition 5** *Let $\mathcal{H}$ be a geometry from Table I or II of rank at least 3 and let H be a group in the row corresponding to $\mathcal{H}$. Suppose that $\sigma : \mathcal{H} \to \overline{\mathcal{H}}$ is a covering of geometries which commutes with the action of H and let $\overline{H}$ denote the action induced by H on $\overline{\mathcal{H}}$. Then the pair $(\overline{\mathcal{H}}, \overline{H})$ is also from Table I or II, respectively.*

**Proof.** Suppose first that $\mathcal{H}$ is not a $P$-geometry of rank 3. Then by Theorem 2 (iv) and Corollary 4 $\mathcal{H}$ is simply connected and $H$ is the only flag-transitive automorphism group of $\mathcal{H}$, in particular $H$ is the group of all liftings of elements of $\overline{H}$ to automorphisms of $\mathcal{H}$. Let $N$ be the kernel of the homomorphism of $H$ onto $\overline{H}$. Then $N$ is the deck group of $\sigma$ and hence $N$ acts regularly on each of the fibers of $\sigma$. So $N = 1$ if and only if $\sigma$ is an isomorphism. It follows from the structure of $H$ that $H/O_3(H)$ is a non-abelian simple group and $O_3(H)$, if non-trivial, is an irreducible $GF(3)$-module for $H/O_3(H)$. Hence either $N = 1$ or $N = O_3(H)$. In the latter case $\overline{\mathcal{H}} = \mathcal{G}(H/O_3(H))$ and by Theorem 2 (iii) the mapping $\mathcal{H} \to \overline{\mathcal{H}}$ is not a covering. Hence $N = 1$. The situation when $\mathcal{H}$ is a $P$-geometry of rank 3 (i.e., $\mathcal{G}(M_{22})$ or $\mathcal{G}(3 \cdot M_{22})$) can treated in a similar way with a few extra possibilities to be considered. □

Below we outline our main strategy for proving Theorem 3. Let $\mathcal{G}$ be a $P$- or $T$-geometry of rank $n \geq 3$, $G$ be a flag-transitive automorphism group of $G$ and

$$\mathcal{A} = \mathcal{A}(G, \mathcal{G}) = \{G_i \mid 1 \leq i \leq n\}$$

be the amalgam of maximal parabolics associated with the action of $G$ on $\mathcal{G}$ (here $G_i = G(x_i)$ is the stabilizer in $G$ of the element $x_i$ of type $i$ in a maximal flag $\Phi = \{x_1, ..., x_n\}$ in $\mathcal{G}$). Our goal is to identify $\mathcal{A}$ up to isomorphism or, more specifically, to show that $\mathcal{A}$ is isomorphic to the amalgam $\mathcal{A}(H, \mathcal{H})$ for a geometry $\mathcal{H}$ and a group $H$ from Table I or II. In fact, it is sufficient to show that given the type of $\mathcal{G}$ and its rank there are at most as many possibilities for the isomorphism type of $\mathcal{A}$ as there are corresponding pairs in Tables I and II.

We proceed by induction on the rank $n$ and assume that all the flag-transitive $P$- and $T$-geometries of rank up to $n-1$ (along with their flag-transitive automorphism groups) are known (as in the tables). Then we can assume that for every $1 \leq i \leq n$ the residue $\mathrm{res}_{\mathcal{G}}(x_i)$ and the action $\overline{G}_i$ of $G_i$ on this residue are known. The kernel $K_i$ of this action is a subgroup in the Borel subgroup $B = \cap_{i=1}^{n} G_i$ which in all the cases turns out to be a 2-group.

The induction hypothesis can be used further since certain normal factors of $K_i$ resemble the structure of the residue $\mathrm{res}_{\mathscr{G}}(x_i)$. The most important case is that the action of $K_1$ on the set of points collinear to $x_1$ is a quotient of the universal representation module of the residue $\mathrm{res}_{\mathscr{G}}(x_1)$, which is a $P$- or $T$-geometry.

Thus, in order to accomplish the identification of the amalgams of maximal parabolics it would be helpful (and essential within our approach) to determine the universal representations of the known $P$- and $T$-geometries. Recall that if $\mathscr{H}$ is a geometry (or rather a point-line incidence system) with three points per line, then the universal representation module $V(\mathscr{H})$ is a group generated by pairwise commuting involutions indexed by the points of $\mathscr{H}$ and subject to the relations that the product of the three involutions corresponding to a line is the identity. It is immediate from the definition that $V(\mathscr{H})$ is an elementary abelian 2-group (possibly trivial).

**Table III. Natural representations of $P$-geometries**

| Rank | Geometry $\mathscr{H}$ | dim $V(\mathscr{H})$ | $R(\mathscr{H})$ |
|---|---|---|---|
| 2 | $\mathscr{G}(Alt_5)$ | 6 | infinite |
| 3 | $\mathscr{G}(M_{22})$ | 11 | $\overline{\mathscr{C}}_{11}$ |
| | $\mathscr{G}(3 \cdot M_{22})$ | 23 | ? |
| 4 | $\mathscr{G}(M_{23})$ | 0 | 1 |
| | $\mathscr{G}(Co_2)$ | 23 | $\overline{\Lambda}^{(23)}$ |
| | $\mathscr{G}(3^{23} \cdot Co_2)$ | 23 | ? |
| | $\mathscr{G}(J_4)$ | 0 | $J_4$ |
| 5 | $\mathscr{G}(BM)$ | 0 | $2 \cdot BM$ |
| | $\mathscr{G}(3^{4371} \cdot BM)$ | 0 | ? |

For the geometries $\mathscr{G}(J_4)$, $\mathscr{G}(BM)$, $\mathscr{G}(M)$ of large sporadic simple groups the universal representation modules are trivial and this is the reason why these geometries do not appear as residues in flag-transitive $P$- and $T$-geometries of higher ranks. On the other hand, if $\mathcal{G}$ is one of the above three geometries and $G$ is the automorphism group of $\mathcal{G}$, then the points and lines of $\mathcal{G}$ are certain elementary abelian subgroups in $G$ of order 2 and $2^2$, respectively, so that the incidence relation is via inclusion. This means that $G$ is a quotient of the universal representation group $R(\mathcal{G})$ of $\mathcal{G}$. The definition of $R(\mathcal{G})$ is that of $V(\mathcal{G})$ with the wording 'pairwise commuting' removed. Since $V(\mathcal{G})$ is the quotient of $R(\mathcal{G})$ over the commutator subgroup of $R(\mathcal{G})$, sometimes it turns out to be easier to show that $R(\mathcal{G})$ is perfect rather than showing the triviality of $V(\mathcal{G})$ directly. In Part I we calculate the modules $V(\mathcal{G})$ for all flag-transitive $P$- and $T$-geometries and the groups $R(\mathcal{G})$ for most of them. These results are summarized in Tables III and IV. The determination problem for $R(\mathcal{G})$ for various geometries $\mathcal{G}$ (including the $P$- and $T$-geometries) is of an independent interest, since, in particular, representations control the $c$-extensions of geometries.

**Table IV. Natural representations of $T$-geometries**

| Rank | Geometry $\mathcal{H}$ | dim $V(\mathcal{H})$ | $R(\mathcal{H})$ |
|---|---|---|---|
| 2 | $\mathcal{G}(3 \cdot S_4(2))$ | 11 | infinite |
| 3 | $\mathcal{G}(M_{24})$ | 11 | $\overline{\mathcal{C}}_{11}$ |
| 4 | $\mathcal{G}(Co_1)$ | 24 | $\overline{\Lambda}^{(24)}$ |
| 5 | $\mathcal{G}(M)$ | 0 | $M$ |
| $n$ | $\mathcal{G}(3^{[\frac{n}{2}]_2} \cdot S_{2n}(2))$ | $(2n+1)+2^n(2^n-1)$ | infinite |

The knowledge of the module $V(\mathcal{H})$ for known geometries $\mathcal{H}$ forms a strong background for the classification of the amalgams $\mathcal{A}(G,\mathcal{G})$ for the flag-transitive automorphism groups $G$ of a $P$- or $T$-geometry $\mathcal{G}$. This classification is presented in Part II of this second volume. As an immediate outcome we have the following.

**Proposition 6** *Let $\mathcal{G}$ be a P- or T-geometry and G be a flag-transitive automorphism group of $\mathcal{G}$. Let p be a point (an element of type 1) in $\mathcal{G}$, $\mathcal{F} = \mathrm{res}_{\mathcal{G}}(p)$, $F = G(p)$ be the stabilizer of p in G and $\overline{F}$ be the action induced by F on $\mathcal{F}$. Then $(\mathcal{F}, \overline{F})$ is not one of the following pairs:*

$$(\mathcal{G}(M_{23}), M_{23}),\quad (\mathcal{G}(BM), BM),\quad (\mathcal{G}(3^{4371}\cdot BM), 3^{4371}\cdot BM),\quad (\mathcal{G}(M), M).$$

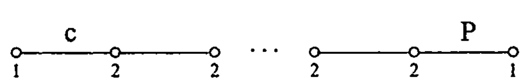
**Proof.** We apply (1.5.2). Suppose that $(\mathcal{F}, \overline{F})$ is one of the above four pairs. The condition (i) in (1.5.2) follows from Tables III and IV. If $(p, l, \pi)$ is a flag of rank 3 in $\mathcal{G}$ consisting of a point $p$, line $l$ and plane $\pi$, then the structure of the maximal parabolics associated with the action of $\overline{F}$ on $\mathcal{F}$ (cf. pp. 114, 224, 210 and 234 in [Iv99]) shows that in each case $\overline{F}(\pi)$ induces $Sym_3$ on the set of lines incident to $p$ and $\pi$ (so that (ii) in (1.5.2) holds) and that $\overline{F}(l)$ is isomorphic respectively to

$$M_{22},\quad 2_+^{1+22}.Co_2,\quad (2_+^{1+22}\times 3^{23}).Co_2,\quad 2_+^{1+24}.Co_1.$$

Since none of these groups contains a subgroup of index 2 the proof follows. □

Notice that in the case $(\mathcal{F}, \overline{F}) = (\mathcal{G}(J_4), J_4)$ the subgroup $\overline{F}(l) \cong 2_+^{1+12}\cdot 3\cdot \mathrm{Aut}\,M_{22}$ does contain a subgroup of index two, so this case requires a further analysis to be eliminated (this will be accomplished in Section 11.6).

The knowledge of universal representations groups enables us to construct and prove simple connectedness of so-called affine $c$-extensions $\mathcal{AF}(\mathcal{G}, R(\mathcal{G}))$ of the known $P$- and $T$-geometries $\mathcal{G}$ (cf. Section 2.7). These extensions have diagrams



or

depending on whether $\mathcal{G}$ is a $P$- or $T$-geometry.

We formulate here the results on both simple connectedness and the full automorphisms groups.

**Proposition 7** *The following assertions hold:*

(i) *$\mathcal{AF}(\mathcal{G}(M_{22}), \mathcal{C}_{11})$ is simply connected with the automorphism group $2^{11} : \mathrm{Aut}\,M_{22}$;*

(ii) *$\mathcal{G}(M_{23})$ does not possess flag-transitive affine c-extensions;*

(iii) $\mathscr{AF}(\mathscr{G}(Co_2), \overline{\Lambda}^{(23)})$ *is simply connected with the automorphism group* $2^{23} : Co_2$;

(iv) $\mathscr{AF}(\mathscr{G}(J_4), J_4)$ *is simply connected with the automorphism group* $J_4 \wr 2$;

(v) $\mathscr{AF}(\mathscr{G}(BM), 2 \cdot BM)$ *is simply connected with the automorphism group* $(2 \cdot BM * 2 \cdot BM).2$;

(vi) $\mathscr{AF}(\mathscr{G}(M_{24}), \overline{\mathscr{C}}_{11})$ *is simply connected with the automorphism group* $2^{11} : M_{24}$;

(vii) $\mathscr{AF}(\mathscr{G}(Co_1), \overline{\Lambda}^{(24)})$ *is simply connected with the automorphism group* $2^{24} : Co_1$;

(viii) $\mathscr{AF}(\mathscr{G}(M), M)$ *is simply connected with the automorphism group* $M \wr 2$ *(the Bimonster).* □

The analysis of the amalgam $\mathscr{A}$ is via consideration of the normal factors of the parabolics $G_1$ and $G_n$. This analysis brings us to a restricted number of possibilities for the normal factors.

We proceed by accomplishing the following sequence of steps (we follow notation as introduced at the end of Section 1.1). First we reconstruct up to isomorphism the point stabilizer $G_1$. Our approach is inductive so we assume that the action $\overline{G}_1 = G_1/K_1$ of $G_1$ on $\mathrm{res}_{\mathscr{G}}(x_1)$ is one of the known actions in Table I or II. Then we turn to $G_2$, or more precisely to the subamalgam $\mathscr{B} = \{G_1, G_2\}$ in $\mathscr{A}$. The subgroup $G_2$ is the stabilizer of the line $x_2$ and it induces $Sym_3$ on the triple of points incident to $x_2$ (of course $x_1$ is in this triple). Hence $G_{12} = G_1 \cap G_2$ contains a subgroup $K_2^-$ of index 2 (the pointwise stabilizer of $x_2$), which is normal in $G_2$ and $G_2/K_2^- \cong Sym_3$. Therefore we identify $K_2^-$ as a subgroup of $G_1$, determine the automorphism group of $K_2^-$ and then classify the extensions of $K_2^-$ by automorphisms forming $Sym_3$. In this step we can refine the choice of the isomorphism type of $G_1$, since within the wrong choice $K_2^-$ might not possess the required automorphisms.

A glance at Tables I and II gives the following.

**Proposition 8** *Let $\mathscr{F}$ be the residue of a point in a (known) P- or T-geometry of rank $n \geq 2$ (so that either $n \geq 3$ and $\mathscr{F}$ is itself a P- or T-geometry or $n = 2$ and $\mathscr{F}$ is of rank 1 with 2 or 3 points, respectively) and let F be a flag-transitive automorphism group of $\mathscr{F}$. Then* $|\mathrm{Aut}\,\mathscr{F} : F| \leq 2$. □

This immediately gives the following

**Proposition 9** *In the above terms $\overline{G}_2 = G_2/K_2$ is isomorphic to a subgroup of index at most 2 in the direct product*

$$G_2/K_2^- \times G_2/K_2^+,$$

*where $G_2/K_2^- \cong Sym_3$ and $G_2/K_2^+$ is a flag-transitive automorphism group of $\mathrm{res}_{\mathscr{G}}^+(x_2)$. In particular the centre of $O^2(G_2/K_2)$ contains a subgroup $X$ which permutes transitively the points incident to $x_2$.* □

By Proposition 9 the automorphisms of $K_2^-$ that we were talking about can always be chosen to commute with $O^2(K_2^-/K_2)$.

Next we extend $\mathscr{B}$ to the rank 3 amalgam $\mathscr{C} = \{G_1, G_2, G_3\}$. Towards this end we first identify $\mathscr{D} = \{G_{13}, G_{23}\}$ as a subamalgam in $\mathscr{B}$. Since the action of $G_1$ on $\mathrm{res}_{\mathscr{G}}(x_1)$ is known, $G_{13}$ and $G_{123}$ are specified uniquely up to conjugation in $G_1$. By Proposition 9, $G_{23} = \langle G_{123}, Y\rangle$, where $Y$ maps onto the subgroup $X$ as in that proposition. Since $K_2$ is a 2-group, we can choose $Y$ to be a Sylow 3-subgroup (of order 3) in $K_2^+$.

Thus we obtain the amalgam $\widetilde{\mathscr{C}} = \{G_1, G_2, \widetilde{G}_3\}$, where $\widetilde{G}_3$ is the universal completion (free amalgamated product) of the subamalgam $\mathscr{D}$ in $\mathscr{B}$. In order to get the amalgam $\mathscr{C}$ we have to identify in $\widetilde{G}_3$ the normal subgroup $N$ such that $G_3 = \widetilde{G}_3/N$. The subgroup $K_3^-$ can be specified as the largest subgroup in $G_{123}$ which is normal in both $G_{13}$ and $G_{23}$. Then

$$G_3/K_3^- \cong L_3(2), \quad G_{13}/K_3^- \cong G_{23}/K_3^- \cong Sym_4$$

and the latter two quotients are maximal parabolics in the former one. In all cases the parabolics are 2-constrained and the images of both $G_{13}$ and $G_{23}$ in $\mathrm{Out}\, K_3^-$ are isomorphic to $Sym_4$. These two images must generate in $\mathrm{Out}\, K_3^-$ the group $L_3(2)$ (otherwise there is no way to extend $\mathscr{B}$ to a correct $\mathscr{C}$). Hence we may assume that

$$\widetilde{G}_3/(K_3^- C_{\widetilde{G}_3}(K_3^-)) \cong L_3(2).$$

Since $\widetilde{G}_3/K_3^- N$ is also $L_3(2)$, we see that $N$ must be a subgroup in the centralizer of $K_3^-$ in $\widetilde{G}_3$, which trivially intersects $K_3^-$ and such that

$$K_3^- N = K_3^- C_{\widetilde{G}_3}(K_3^-).$$

The easiest situation is when the centre of $K_3^-$ is trivial in which case we are forced to put $N = C_{\widetilde{G}_3}(K_3^-)$, so that $N$ is uniquely determined (8.5.1). In fact the uniqueness of $N$ can be proved under a weaker assumption: the centre of $K_3^-$ does not contain 8-dimensional composition factors with respect to $\widetilde{G}_3/K_3^- C_{\widetilde{G}_3}(K_3^-) \cong L_3(2)$ (8.5.3). The following property of the known $P$- and $T$-geometries (which can easily be checked by inspection using information contained in [Iv99] and [IMe99]) shows that (8.5.3) always applies when $\mathscr{B}$ is isomorphic to the amalgam from a known example.

**Proposition 10** *Let* $(\mathscr{H}, H)$ *be a pair from Table I or II and suppose that the rank of* $\mathscr{H}$ *is at least* 3*. Let* $\pi$ *be a plane in* $\mathscr{H}$ *(an element of type* 3*),* $H(\pi)$ *be the stabilizer of* $\pi$ *in* $H$ *and* $K^-(\pi)$ *be the kernel of the action of* $H(\pi)$ *on the set of points and lines incident to* $\pi$ *(these points and lines form a projective plane of order* 2*). Then every chief factor of* $H(\pi)$ *inside* $Z(K^-(\pi))$ *is an elementary abelian* 2*-group which is either* 1*- or* 3*-dimensional module for* $H(\pi)/K^-(\pi) \cong L_3(2)$. □

After $\mathscr{C}$ is reconstructed, the structure of the whole amalgam $\mathscr{A}$ is pretty much forced. Indeed $G_4$ is a completion of the subamalgam $\mathscr{E} = \{G_{i4} \mid 1 \le i \le 3\}$ in $\mathscr{C}$. It turns out that this subamalgam is always uniquely determined in $\mathscr{C}$ (up to conjugation). On the other hand, the residue $\mathrm{res}^-_{\mathscr{G}}(x_4)$ is the rank 3 projective $GF(2)$-geometry, which is simply connected. By the fundamental principle (1.4.6) this implies that $G_4$ is the universal completion of $\mathscr{E}$. Hence there is a unique way to extend $\mathscr{C}$ to the rank 4 amalgam and to carry on in the same manner to get the whole amalgam $\mathscr{A}$ of maximal parabolics.

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