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A. A. Ivanov and S. V. Shpectorov
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1

Preliminaries

In this introductory chapter after recalling the main notions and notation concerning diagram geometries and their flag-transitive automorphism groups we prove the fundamental principle (Theorem 1.4.5), which relates the universal cover of a geometry $\mathcal{G}$ and the universal completion of the amalgam $\mathcal{A}$ of maximal parabolics in a flag-transitive automorphism group G of $\mathcal{G}$. This principle lies in the foundation of our approach to the classification of flag-transitive geometries in terms of their diagrams. In the last section of the chapter we recall what is meant by a representation of geometry. The importance of representations for our classification approach is explained in Proposition 1.5.1, which shows that under certain natural assumptions one of the chief factors of the stabilizer of a point in a flag-transitive automorphism group carries a representation of the residue of the point (this result is generalized in Proposition 9.4.1 for other maximal parabolics).

1.1 Geometries and diagrams

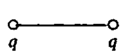
In this section we recall the main terminology and notations concerning diagram geometries (cf. Introduction in [Iv99] and references therein).

An *incidence system* of *rank* n is a set $\mathcal{G}$ of *elements* that is a disjoint union of subsets $\mathcal{G}^{\alpha_1}, ..., \mathcal{G}^{\alpha_n}$ (where $\mathcal{G}^{\alpha_i}$ is the set of elements of type α_i in $\mathcal{G}$) and a binary reflexive symmetric *incidence relation* on $\mathcal{G}$, with respect to which no two distinct elements of the same type are incident. We can identify $\mathcal{G}$ with its *incidence graph* $\Gamma = \Gamma(\mathcal{G})$ having $\mathcal{G}$ as the set of vertices, in which two distinct elements are adjacent if they are incident. A *flag* in $\mathcal{G}$ is a set Φ of pairwise incident elements (the vertex-set of a complete subgraph in the incidence graph). The *type* (respectively *cotype*) of Φ is the set of types in $\mathcal{G}$ present (respectively not present) in Φ. The sizes of these sets are the *rank* and the *corank* of Φ. By the definition a flag contains at most one element of any given type. If Φ is a flag in $\mathcal{G}$, then the *residue* $\mathrm{res}_{\mathcal{G}}(\Phi)$ of Φ in $\mathcal{G}$ is an incidence system whose elements are those from $\mathcal{G} \setminus \Phi$ incident to every element in Φ with respect to the induced type function and incidence relation.

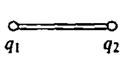
An incidence system $\mathcal{G}$ of rank n is called a *geometry* if for every flag Φ (possibly empty) of corank at least 2 and every $\alpha_i \neq \alpha_j$ from the cotype of Φ the subgraph in the incidence graph induced by $\mathcal{G}^{\alpha_i} \cap \mathcal{G}^{\alpha_j} \cap \mathrm{res}_{\mathcal{G}}(\Phi)$ is non-empty and connected (this implies that a maximal flag contains elements of all types). Clearly the residue of a geometry is again a geometry.

In what follows, unless stated otherwise, the set of types in a geometry of rank n is taken to be $\{1, 2, ..., n\}$. A diagram of a geometry $\mathcal{G}$ is a graph with labeled edges on the set of types in $\mathcal{G}$ in which the edge (or absence of such) joining i and j symbolizes the class of geometries appearing as residues of flags of cotype $\{i, j\}$ in $\mathcal{G}$. Under the node i it is common to write the number q_i such that every flag of cotype i in $\mathcal{G}$ is contained in exactly $q_i + 1$ maximal flags. We will mainly deal with the following rank 2 residues:

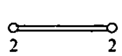
- generalized digon: any two elements of different types are incident, the incidence graph is complete bipartite with parts of size $q_1 + 1$ and $q_2 + 1$;



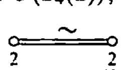
- projective plane $pg(2, q)$ of order q;



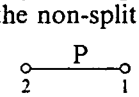
- generalized quadrangle $gq(q_1, q_2)$ of order (q_1, q_2);



- the generalized quadrangle $\mathcal{G}(S_4(2))$ of order $(2, 2)$, whose elements are the 2-element subsets of a 6-set and the partitions of the 6-set three 2-element subsets (equivalently the 1-subspaces and totally isotropic 2-subspaces in a 4-dimensional symplectic $GF(2)$-space) with the natural incidence relation; the automorphism group is $S_4(2) \cong Sym_6$ and the outer automorphism of this group induces a diagram automorphism of $\mathcal{G}(S_4(2))$;



- the triple cover $\mathcal{G}(3 \cdot S_4(2))$ of $\mathcal{G}(S_4(2))$ associated with the non-split extension $3 \cdot S_4(2) \cong 3 \cdot Sym_6$;



- the geometry $\mathcal{G}(Alt_5)$ of edges and vertices of the Petersen graph; the vertices of the Petersen graph are the 2-element subsets of a 5-set and two such subsets are adjacent if they are disjoint;

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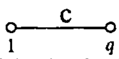

- the geometry of 1- and 2-element subsets of a $(q+2)$-set with the incidence relation defined by inclusion; when $q = 2$ this is the affine plane of order 2.

If Φ is a flag in $\mathcal{G}$, then the diagram of $\mathrm{res}_{\mathcal{G}}(\Phi)$ is the subdiagram in the diagram of $\mathcal{G}$ induced by the cotype of Φ.

The notation we are about to introduce can be applied to any rank n geometry $\mathcal{G}$, but it is particularly useful when $\mathcal{G}$ belongs to a string diagram, i.e., when the residue of a flag of cotype $\{i, j\}$ is a generalized digon whenever $|i - j| \geq 2$.

For an element x_i of type i, where $1 \leq i \leq n$, we denote by $\mathrm{res}^+_{\mathcal{G}}(x_i)$ and $\mathrm{res}^-_{\mathcal{G}}(x_i)$ the set of elements of types larger than i and less than i, respectively, that are incident to x_i. When $\mathcal{G}$ belongs to a string diagram they are residues of a flag of type $\{1, ..., i\}$ containing x_i and of a flag of type $\{i, ..., n\}$ containing x_i, respectively. If G is an automorphism group of $\mathcal{G}$ (often assumed to be flag-transitive), then $G(x_i)$ is the stabilizer of x_i in G, $K(x_i)$, $K^+(x_i)$ and $K^-(x_i)$ are the kernels of the actions of $G(x_i)$ on $\mathrm{res}_{\mathcal{G}}(x_i)$, $\mathrm{res}^+_{\mathcal{G}}(x_i)$ and $\mathrm{res}^-_{\mathcal{G}}(x_i)$, respectively. By $L(x_i)$ we denote the kernel of the action of $G(x_i)$ on the set of elements y_i of type i in $\mathcal{G}$ such that there exists a premaximal flag Ψ of cotype i such that both $\Psi \cup \{x_i\}$ and $\Psi \cup \{y_i\}$ are maximal flags.

When we deal with a fixed maximal flag $\Phi = \{x_1, ..., x_n\}$ in $\mathcal{G}$, we write G_i instead of $G(x_i)$, K_i instead of $K(x_i)$, etc. If $J \subseteq \{1, 2, ..., n\}$, then

$$G_J = \bigcap_{j \in J} G_j$$

and we write, for instance, G_{12} instead of $G_{\{1,2\}}$, and similar. The subgroups G_J are called *parabolic subgroups* or simply *parabolics*. The subgroups G_i are *maximal parabolics*. Most of our geometries are 2-local, so that the parabolics are 2-local subgroups and we put $Q(x_i) = O_2(G(x_i))$ (which can also be written simply as Q_i). Notice that if $\mathcal{G}$ belongs to a string diagram and x_1 is a point then L_1 is the elementwise stabilizer in G_1 of the set of points collinear to x_1.

1.2 Coverings of geometries

Let $\mathcal{H}$ and $\mathcal{G}$ be geometries (or more generally incidence systems). A *morphism* of geometries is a mapping $\varphi : \mathcal{H} \to \mathcal{G}$ of the element set of $\mathcal{H}$ the element set of $\mathcal{G}$ which maps incident pairs of elements onto incident pairs and preserves the type function. A bijective morphism, whose inverse is also a morphism is called an *isomorphism*.

A surjective morphism $\varphi : \mathcal{H} \to \mathcal{G}$ is said to be a *covering* of $\mathcal{G}$ if for every non-empty flag Φ of $\mathcal{H}$ the restriction of φ to the residue $\mathrm{res}_{\mathcal{H}}(\Phi)$ is an isomorphism onto $\mathrm{res}_{\mathcal{G}}(\varphi(\Phi))$. In this case $\mathcal{H}$ is a *cover* of $\mathcal{G}$ and $\mathcal{G}$ is a *quotient* of $\mathcal{H}$. If every covering of $\mathcal{G}$ is an isomorphism then $\mathcal{G}$ is said to be *simply connected.* Clearly a morphism is a covering if its restriction to the residue of every element (considered as a flag of rank 1) is an isomorphism. If $\psi : \widetilde{\mathcal{G}} \to \mathcal{G}$ is a covering and $\widetilde{\mathcal{G}}$ is simply connected, then ψ is the *universal covering* and $\widetilde{\mathcal{G}}$ is the *universal cover* of $\mathcal{G}$. The universal cover of a geometry exists and it is uniquely determined up to isomorphism. If $\varphi : \mathcal{H} \to \mathcal{G}$ is any covering then there exists a covering $\chi : \widetilde{\mathcal{G}} \to \mathcal{H}$ such that ψ is the composition of χ and φ.

A morphism $\varphi : \mathcal{H} \to \mathcal{G}$ of arbitrary incidence systems is called an *s-covering* if it is an isomorphism when restricted to every residue of rank s or more. This means that if Φ is a flag whose corank is less than or equal to s, then the restriction of φ to $\mathrm{res}_{\mathcal{H}}(\Phi)$ is an isomorphism. An incidence system, every s-cover of which is an isomorphism, is said to be *s-simply connected*. It is clear that when $s = n - 1$'s-covering' and 'covering' mean the same thing.

An isomorphism of a geometry onto itself is called an *automorphism*. By the definition an isomorphism preserves the types. Sometimes we will need a more general type of automorphisms which permute types. We will refer to them as *diagram automorphisms*.

The set of all automorphisms of a geometry $\mathcal{G}$ forms a group called the *automorphism group* of $\mathcal{G}$ and denoted by $\mathrm{Aut}\,\mathcal{G}$. An automorphism group G of $\mathcal{G}$ (that is a subgroup of $\mathrm{Aut}\,\mathcal{G}$) is said to be *flag-transitive* if any two flags Φ_1 and Φ_2 in $\mathcal{G}$ of the same type are in the same G-orbit. Clearly an automorphism group is flag-transitive if and only if it acts transitively on the set of maximal flags in $\mathcal{G}$. A geometry $\mathcal{G}$ possessing a flag-transitive automorphism group is said to be *flag-transitive*.

Let $\varphi : \mathcal{H} \to \mathcal{G}$ be a covering and H be a group of automorphisms of $\mathcal{H}$. We say that H commutes with φ if for every $h \in H$ whenever $\varphi(x) = \varphi(y)$, for $x, y \in \mathcal{H}$, the equality $\varphi(x^h) = \varphi(y^h)$ holds. In this case we can define the action of h on $\mathcal{G}$ via $\varphi(x)^h = \varphi(x^h)$. Let the induced action be denoted by $\bar{H}$. The kernel of the action is called the subgroup of *deck transformation* in H with respect to φ.

The following observation is quite important.

Lemma 1.2.1 *Let $\varphi : \mathcal{H} \to \mathcal{G}$ be a covering of geometries and H be a flag-transitive automorphism group of $\mathcal{H}$ commuting with φ. Then the action $\bar{H}$ induced by H on $\mathcal{G}$ is flag-transitive.* □

Let $\mathcal{G}$ be a geometry (or rather an incidence system) of rank n and N be a group of automorphisms of $\mathcal{G}$. Then the *quotient of* $\mathcal{G}$ *over* N is an incidence system $\bar{\mathcal{G}}$ whose elements of type i are the orbits of N on $\mathcal{G}^i$ and two N-orbits, say Ω and Δ, are incident if some $\omega \in \Omega$ is incident to some $\delta \in \Delta$ in $\mathcal{G}$. If the mapping $\varphi : \mathcal{G} \to \bar{\mathcal{G}}$ that sends every element $x \in \mathcal{G}$ onto its N-orbit, is a covering and N is normal in H then it is easy to see that H commutes with φ.

1.3 Amalgams of groups

Our approach for classifying P- and T-geometry is based on the method of group amalgams. This method can be applied to the classification of other geometries in terms of their diagrams and already has been proved to be adequate, for instance within the classification of c-extensions of classical dual polar spaces [Iv97], [Iv98].

Let us recall the definition of amalgam and related notions briefly introduced in volume 1 [Iv99]. Here we make our notation slightly more explicit and general.

Definition 1.3.1 *An amalgam* $\mathscr{A}$ *of finite type and rank* $n \geq 2$ *is a set such that for every* $1 \leq i \leq n$ *there is a subset* A_i *in* $\mathscr{A}$ *and a binary operation* $\star_i$ *on* A_i *such that the following conditions hold:*

(A1) $(A_i, \star_i)$ *is a group for* $1 \leq i \leq n$*;*
(A2) $\mathscr{A} = \cup_{i=1}^{n} A_i$*;*
(A3) $|A_i \cap A_j|$ *is finite if* $i \neq j$ *and* $\cap_{i=1}^{n} A_i \neq \emptyset$*;*
(A4) $(A_i \cap A_j, \star_i)$ *is a subgroup in* $(A_i, \star_i)$ *for all* $1 \leq i, j \leq n$*;*
(A5) *if* $x, y \in A_i \cap A_j$ *then* $x \star_i y = x \star_j y$*.*

Abusing the notation we often write $\mathscr{A} = \{A_i \mid 1 \leq i \leq n\}$ in order to indicate explicitly which groups constitute $\mathscr{A}$. In what follows, unless explicitly stated otherwise, all amalgams under consideration will be of finite type.

Let $\mathscr{A} = \{A_i \mid 1 \leq i \leq n\}$ be an amalgam. A *completion* of $\mathscr{A}$ is a pair (G, φ) where G is a group and φ is a mapping of $\mathscr{A}$ into G such that

(C1) G is generated by the image of φ;
(C2) for every i the restriction of φ to A_i is a homomorphism, i.e.,

$$\varphi(x \star_i y) = \varphi(x) \cdot \varphi(y) \quad \text{for all} \quad x, y \in A_i$$

(here '$\cdot$' stands for the group multiplication in G).

If (G_1, φ_1) and (G_2, φ_2) are two completions of the same amalgam $\mathscr{A}$ then a homomorphism χ of G_1 onto G_2 is said to be a *homomorphism of completions* if φ_2 is the composition of φ_1 and χ, i.e., if $\varphi_2(x) = \chi(\varphi_1(x))$ for all $x \in \mathscr{A}$. If K is the kernel of χ then (G_2, φ_2) is called the *quotient* of (G_1, φ_1) over K. Since G_2 is isomorphic to G_1/K via isomorphism $\varphi_2(x) = \varphi_1(x)K$ for $x \in \mathscr{A}$, the completion (G_2, φ_2) is determined by (G_1, φ_1) and K.

When the mapping φ is irrelevant or clear from the context we will talk about a completion G of $\mathscr{A}$. The completion (G, φ) is said to be *faithful* if φ is injective.

Two elements $x, y \in \mathscr{A}$ are said to be *conjugate* in $\mathscr{A}$ if there is a sequence $x_0 = x, x_1, ..., x_m = y$ of elements of $\mathscr{A}$ such that for every $1 \leq j \leq m$ the elements x_{j-1} and x_j are contained in A_i (where i might depend on j) and are conjugate in A_i (in the sense that $x_i = z^{-1}x_{i-1}z$ for some $z \in A_i$). It is easy to see that if (G, φ) is a completion of $\mathscr{A}$ then $\varphi(x)$ and $\varphi(y)$ are conjugate in G whenever x and y are conjugate in $\mathscr{A}$.

For an amalgam $\mathscr{A} = \{A_i \mid 1 \leq i \leq n\}$ let $U(\mathscr{A})$ be the group defined by the following presentation:

$$U(\mathscr{A}) = \langle u_x, x \in \mathscr{A} \mid u_x u_y = u_z \text{ if } x, y, z \in A_i \text{ for some } i \text{ and } x \star_i y = z \rangle.$$

Thus the generators of $U(\mathscr{A})$ are indexed by the elements of $\mathscr{A}$ and the relations are all the equalities that can be seen in the groups constituting the amalgam.

Lemma 1.3.2 *In the above terms let ν be the mapping of $\mathscr{A}$ into $U(\mathscr{A})$ defined by*

$$\nu : x \to u_x$$

for all $x \in \mathscr{A}$. Then $(U(\mathscr{A}), \nu)$ is a completion of $\mathscr{A}$, which is universal in the sense that every completion of $\mathscr{A}$ is a quotient of $(U(\mathscr{A}), \nu)$.

Proof. The fact that $(U(\mathscr{A}), \nu)$ is a completion follows directly from the definition. Let (G, φ) be any completion of $\mathscr{A}$. Define ψ to be a mapping which sends u_x onto $\varphi(x)$ for every $x \in \mathscr{A}$. We claim that ψ extends uniquely to a homomorphism of $U(\mathscr{A})$ onto G. By (C1) ψ maps a generating set of $U(\mathscr{A})$ onto a generating set of G which implies the uniqueness. Now consider a defining relation $u_x u_y = u_z$ of $U(\mathscr{A})$. Then $x, y, z \in A_i$ for some i and $x \star_i y = z$. Since (G, φ) is a representation, we have

$$\psi(u_x)\psi(u_y) = \varphi(x)\varphi(y) = \varphi(z) = \psi(u_z).$$

Hence ψ extends to a homomorphism. □

Thus there is a natural bijection between the completions of $\mathscr{A}$ and the normal subgroups of the universal completion (group) $U(\mathscr{A})$. If N is a normal subgroup in $U(\mathscr{A})$ then the corresponding completion is the quotient of $(U(\mathscr{A}), \nu)$ over N. The following result is rather obvious.

Lemma 1.3.3 *An amalgam $\mathscr{A}$ possesses a faithful completion if and only if its universal completion is faithful.* □

The subgroup $B := \cap_{i=1}^{n} A_i$ is called the *Borel subgroup* of $\mathscr{A}$. By (A3) and (A5), B is a finite group in which the group operation coincides with the restriction of $\star_i$ for every $1 \leq i \leq n$. In particular, the identity element of B is the identity element of every $(A_i, \star_i)$. The following result can be easily deduced from Section 35 in [Kur60].

Proposition 1.3.4 *Let $\mathscr{A} = \{A_i \mid 1 \leq i \leq n\}$ be an amalgam of rank $n \geq 2$ with Borel subgroup B. Suppose that $B = A_i \cap A_j$ for all $1 \leq i < j \leq n$ (which always holds when $n = 2$) and $\mathscr{A} \nsubseteq A_i$ for every $1 \leq i \leq n$. Then the universal completion of $\mathscr{A}$ is faithful and $U(\mathscr{A})$ is the free amalgamated product of the groups A_i over the subgroup B, in particular, it is infinite.* □

One should not confuse the set of all amalgams and their very special class covered by (1.3.4). For an amalgam $\mathscr{A}$ of rank $n \geq 3$ the universal completions might or might not be faithful and might be infinite or finite (or even trivial). In general it is very difficult to decide what $U(\mathscr{A})$ is and this problem is clearly equivalent to the identification problem of a group defined by generators and relations.

A subgroup M of B which is normal in $(A_i, \star_i)$ for every $1 \leq i \leq n$ is said to be a *normal subgroup* of the amalgam $\mathscr{A}$. The largest normal subgroup in $\mathscr{A}$ is called the *core* of $\mathscr{A}$ and the amalgam is said to be *simple* if its core is trivial (the identity subgroup of B). Notice that if M is normal in $\mathscr{A}$ then $\varphi(M)$ is a normal subgroup in G for every completion (G, φ) of $\mathscr{A}$, but even when $\mathscr{A}$ is a simple amalgam, a completion group G is not necessarily simple.

1.4 Simple connectedness via universal completion

Let $\mathscr{G}$ be a geometry of rank n, G be a flag-transitive automorphism group of $\mathscr{G}$ and $\Phi = \{x_1, ..., x_n\}$ be a maximal flag in $\mathscr{G}$, where x_i is of type i. Let $G_i = G(x_i)$ be the stabilizer of x_i in G (the maximal parabolic of type i associated with the action of G on $\mathcal{G}$) and

$$\mathscr{A} := \mathscr{A}(G,\mathcal{G}) = \{G_i \mid 1 \leq i \leq n\}$$

be the amalgam of the maximal parabolics.

We define the *coset geometry* $\mathscr{C} = \mathscr{C}(G,\mathscr{A})$ in the following way (it might not be completely obvious at this stage that $\mathscr{C}$ is a geometry rather than just an incidence system). The elements of type i in $\mathscr{C}$ are the right cosets of the subgroup G_i in G, so that

$$\mathscr{C}^i = \{G_i g \mid g \in G\} \quad \text{and}$$

$$\mathscr{C} = \bigcup_{1 \leq i \leq n} \mathscr{C}^i \quad \text{(disjoint union).}$$

Two different cosets are incident if and only if they have an element in common:

$$G_i h \sim G_j k \iff G_i h \cap G_j k \neq \emptyset.$$

Lemma 1.4.1 *Let ϱ be the mapping which sends the coset $G_i g$ from $\mathscr{C}^i$ onto the image x_i^g of x_i under $g \in G$:*

$$\varrho : G_i g \mapsto x_i^g.$$

Then ϱ is an isomorphism of $\mathscr{C}$ onto $\mathcal{G}$.

Proof. First notice that ϱ is well defined, since if $g' \in G_i g$, say $g' = fg$ for $f \in G_i$, then we have

$$x_i^{g'} = x_i^{fg} = (x_i^f)^g = x_i^g.$$

This also shows that for $y_i \in \mathcal{G}^i$ the set $\varrho^{-1}(y_i)$ consists of the elements of G which map x_i onto y_i.

Next we check that ϱ preserves the incidence relation. Suppose first that $G_i h$ and $G_j k$ are incident in $\mathscr{C}$, which means that they contain an element g in common. Then $G_i h = G_i g$, $G_j k = G_j g$ and

$$\{\varrho(G_i h), \varrho(G_j k)\} = \{x_i^g, x_j^g\}.$$

Since x_i and x_j are incident and g is an automorphism of $\mathcal{G}$, x_i^g and x_j^g are also incident. On the other hand, suppose that $y_i = \varrho(G_i h)$ and $y_j = \varrho(G_j k)$ are incident elements of types i and j in $\mathcal{G}$. Since G acts flag-transitively on $\mathcal{G}$, there is a $g \in G$ such that $\{y_i, y_j\} = \{x_i^g, x_j^g\}$. By the above observation $g \in G_i h \cap G_j k$, which means that $G_i h$ and $G_j k$ are incident in $\mathscr{C}$. □

In the above terms, for $1 \leq i \leq n$ the maximal parabolic G_i acts flag-transitively on the residue $\mathrm{res}_{\mathscr{G}}(x_i)$ of x_i in $\mathscr{G}$. By (1.4.1) we have the following.

Corollary 1.4.2 *The residue* $\mathrm{res}_{\mathscr{G}}(x_i)$ *is isomorphic to the coset geometry* $\mathscr{C}(G_i, \mathscr{A}_i)$, *where*

$$\mathscr{A}_i = \{G_i \cap G_j \mid 1 \leq j \leq n, j \neq i\}.$$

□

By the above corollary the isomorphism types of the residues in $\mathscr{G}$ are completely determined by the amalgam $\mathscr{A}$ of maximal parabolics in a flag-transitive automorphism group. Next we discuss up to what extent the amalgam $\mathscr{A}$ determines the structure of the whole of $\mathscr{G}$.

Let $\mathscr{G}$ and $\mathscr{G}'$ be geometries of rank n with flag-transitive automorphism groups G and G', amalgams $\mathscr{A}$ and $\mathscr{A}'$ of maximal parabolics associated with maximal flags $\Phi = \{x_1, ..., x_n\}$ and $\Phi' = \{x_1', ..., x_n'\}$, respectively. Suppose there is an isomorphism $\tau_{\mathscr{A}}$ of $\mathscr{A}'$ onto $\mathscr{A}$ (which maps $G_i' = G'(x_i')$ onto $G_i = G(x_i)$). Suppose first that $\tau_{\mathscr{A}}$ is a restriction to $\mathscr{A}'$ of a homomorphism τ_G of G' onto G. Then τ_G induces a mapping $\tau_{\mathscr{C}}$ of $\mathscr{C}' = \mathscr{C}(G', \mathscr{A}')$ (isomorphic to $\mathscr{G}'$) onto $\mathscr{C} = \mathscr{C}(G, \mathscr{A})$ (isomorphic to $\mathscr{G}$):

$$\tau_{\mathscr{C}} : G_i'g' \mapsto G_i\tau_G(g')$$

for all $1 \leq i \leq n$ and $g' \in G'$.

Lemma 1.4.3 *The mapping* $\tau_{\mathscr{C}}$ *is a covering of geometries.*

Proof. By the definition $\tau_{\mathscr{C}}$ preserves the type function. If $G_i'h'$ and $G_j'k'$ are incident (contain a common element g', say) then their images both contain the element $\tau_G(g')$ and hence they are incident as well. Thus $\tau_{\mathscr{C}}$ is a morphism of geometries. By (1.4.2) and the flag-transitivity of G', $\tau_{\mathscr{C}}$ maps the residue of x' in $\mathscr{G}'$ onto the residue of $\tau_{\mathscr{C}}(x')$ in $\mathscr{G}$ and the proof follows. □

In the above terms G and G' are two completions of the same amalgam $\mathscr{A} \cong \mathscr{A}'$. In general one cannot guarantee that one of the completions is a homomorphic image of the other. But this can be guaranteed if one of the completions is universal.

With G and $\mathscr{A}$ as above, let $\widetilde{G} = U(\widetilde{\mathscr{A}})$ be the universal completion of an amalgam $\widetilde{\mathscr{A}} = \{\widetilde{G}_i \mid 1 \leq i \leq n\}$ and suppose that $\widetilde{\mathscr{A}}$ possesses an isomorphism $\widetilde{\tau}_{\mathscr{A}}$ onto $\mathscr{A}$. Since $\widetilde{G}$ is a universal completion of $\widetilde{\mathscr{A}}$

by (1.4.3) the geometry $\widetilde{\mathcal{G}} := \mathcal{C}(\widetilde{G}, \widetilde{\mathcal{A}})$ possesses a covering $\widetilde{\tau}_{\mathcal{G}}$ onto $\mathcal{G} = \mathcal{C}(G, \mathcal{A})$. We formulate this in the following lemma.

Lemma 1.4.4 *Let G be a faithful completion of the amalgam $\mathcal{A}$. Then there is a covering of $\widetilde{\mathcal{G}} = \mathcal{C}(\widetilde{G}, \widetilde{\mathcal{A}})$ onto $\mathcal{C}(G, \mathcal{A})$.* □

The following result was established independently in [Pasi85], [Ti86] and in an unpublished manuscript by the second author of the present book (who claims that the first author lost it) dated around 1984.

Theorem 1.4.5 *The covering $\widetilde{\tau}_{\mathcal{G}}$ is universal.*

Proof. Let

$$\widehat{\tau} : \widehat{\mathcal{G}} \to \mathcal{G}$$

be the universal covering. Let $\widehat{\Phi} = \{\widehat{x}_1, ..., \widehat{x}_n\}$ be a maximal flag in $\widehat{\mathcal{G}}$ being mapped under $\widehat{\tau}$ onto the maximal flag $\Phi = \{x_1, ..., x_n\}$ in $\mathcal{G}$ (i.e., $\widehat{\tau}(\widehat{x}_i) = x_i$ for $1 \leq i \leq n$).

For $g \in G_i$ let us define an automorphism $\widehat{g} = \widehat{g}^{(i)}$ of $\widehat{\mathcal{G}}$ as follows. First $\widehat{x_i^g} = \widehat{x}_i$. Next, if $\widehat{x} \in \widehat{\mathcal{G}}$ is arbitrary, in order to define $\widehat{x^g}$ we proceed in the following way. Consider a path

$$\widehat{\gamma} = (\widehat{y}_0 = \widehat{x}_i, \widehat{y}_1, ..., \widehat{y}_m = \widehat{x})$$

in $\widehat{\mathcal{G}}$ joining $\widehat{x}_i$ with $\widehat{x}$ (such a path exists since $\widehat{\mathcal{G}}$ is connected). Let

$$\gamma = (y_0 = x_i, y_1, ..., y_m)$$

be the image of $\widehat{\gamma}$ under $\widehat{\tau}$ (i.e., $y_j = \widehat{\tau}(\widehat{y}_j)$ for $0 \leq j \leq m$) and let

$$\gamma^g = (y_0^g = y_0 = x_i, y_1^g, ..., y_m^g)$$

be the image of γ under the element g. Then, since γ^g is a path starting at x_i, there is a unique path

$$\widehat{\gamma^g} = (\widehat{y_0^g} = \widehat{y}_0 = \widehat{x}_i, \widehat{y_1^g}, ..., \widehat{y_m^g})$$

in $\widehat{\mathcal{G}}$ starting at $\widehat{x}_i$ and being mapped onto γ^g under $\widehat{\tau}$. We define $\widehat{x^g}$ to be the end term of $\widehat{\gamma^g}$ (i.e., $\widehat{y_m^g}$ in the above terms). First we show that $\widehat{g}$ is well defined, which means it is independent on the particular choice of the path $\widehat{\gamma}$ joining $\widehat{x}_i$ and $\widehat{x}$. Suppose that $\widehat{\gamma}$ and $\widehat{\delta}$ are paths both starting at $\widehat{x}_i$ and ending at $\widehat{x}$. Then, by a theorem from algebraic topology [Sp66], since $\widehat{\tau}$ is universal, the corresponding images γ and δ are homotopic. Since g is an automorphism of $\mathcal{G}$, it maps the pairs of homotopic paths onto the pairs of homotopic paths. Hence γ^g and δ^g