Special Functions

Special functions, which include the trigonometric functions, have been used for centuries. Their role in the solution of differential equations was exploited by Newton and Leibniz, and the subject of special functions has been in continuous development ever since. In just the past thirty years several new special functions and applications have been discovered.

This treatise presents an overview of the area of special functions, focusing primarily on the hypergeometric functions and the associated hypergeometric series. It includes both important historical results and recent developments and shows how these arise from several areas of mathematics and mathematical physics. Particular emphasis is placed on formulas that can be used in computation.

The book begins with a thorough treatment of the gamma and beta functions, which are essential to understanding hypergeometric functions. Later chapters discuss Bessel functions, orthogonal polynomials and transformations, the Selberg integral and its applications, spherical harmonics, q-series, partitions, and Bailey chains.

This clear, authoritative work will be a lasting reference for students and researchers in number theory, algebra, combinatorics, differential equations, mathematical computing, and mathematical physics.

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To Leonard Carlitz, Om Prakash Juneja,
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Preface

Paul Turán once remarked that special functions would be more appropriately labeled "useful functions." Because of their remarkable properties, special functions have been used for centuries. For example, since they have numerous applications in astronomy, trigonometric functions have been studied for over a thousand years. Even the series expansions for sine and cosine (and probably the arc tangent) were known to Madhava in the fourteenth century. These series were rediscovered by Newton and Leibniz in the seventeenth century. Since then, the subject of special functions has been continuously developed, with contributions by a host of mathematicians, including Euler, Legendre, Laplace, Gauss, Kummer, Eisenstein, Riemann, and Ramanujan.

In the past thirty years, the discoveries of new special functions and of applications of special functions to new areas of mathematics have initiated a resurgence of interest in this field. These discoveries include work in combinatorics initiated by Schützenberger and Foata. Moreover, in recent years, particular cases of long familiar special functions have been clearly defined and applied as orthogonal polynomials.

As a result of this prolific activity and long history one is pulled different directions when writing a book on special functions. First, there are important results from the past that must be included because they are so useful. Second, there are recent developments that should be brought to the attention of those who could use them. One also would wish to help educate the new generation of mathematicians and scientists so that they can further develop and apply this subject. We have tried to do all this, and to include some of the older results that seem to us to have been overlooked. However, we have slighted some of the very important recent developments because a book that did them justice would have to be much longer. Fortunately, specialized books dealing with some of these developments have recently appeared: Petkovšek, Wilf, and Zeilberger [1996], Macdonald [1995], Heckman and Schlicktkrull [1994], and Vilenkin and Klimyk

[1992]. Additionally, I. G. Macdonald is writing a new book on his polynomials in several variables and A. N. Kirillov is writing on R-matrix theory.

It is clear that the amount of knowledge about special functions is so great that only a small fraction of it can be included in one book. We have decided to focus primarily on the best understood class of functions, hypergeometric functions, and the associated hypergeometric series. A hypergeometric series is a series Σa_n with a_{n+1}/a_n a rational function of n. Unfortunately, knowledge of these functions is not as widespread as is warranted by their importance and usefulness. Most of the power series treated in calculus are hypergeometric, so some facts about them are well known. However, many mathematicians and scientists who encounter such functions in their work are unaware of the general case that could simplify their work. To them a Bessel function and a parabolic cylinder function are types of functions different from the $3 - j$ or $6 - j$ symbols that arise in quantum angular momentum theory. In fact these are all hypergeometric functions and many of their elementary properties are best understood when considered as such.

Several important facts about hypergeometric series were first found by Euler and an important identity was discovered by Pfaff, one of Gauss's teachers. However, it was Gauss himself who fully recognized their significance and gave a systematic account in two important papers, one of which was published posthumously. One reason for his interest in these functions was that the elementary functions and several other important functions in mathematics are expressible in terms of hypergeometric functions. A half century after Gauss, Riemann developed hypergeometric functions from a different point of view, which made available the basic formulas with a minimum of computation. Another approach to hypergeometric functions using contour integrals was presented by the English mathematician E. W. Barnes in the first decade of this century. Each of these different approaches has its advantages.

Hypergeometric functions have two very significant properties that add to their usefulness: They satisfy certain identities for special values of the function and they have transformation formulas. We present many applications of these properties. For example, in combinatorial analysis hypergeometric identities classify single sums of products of binomial coefficients. Further, quadratic transformations of hypergeometric functions give insight into the relationship (known to Gauss) of elliptic integrals to the arithmetic-geometric mean. The arithmetic-geometric mean has recently been used to compute π to several million decimal places, and earlier it played a pivotal role in Gauss's theory of elliptic functions.

The gamma function and beta integrals dealt with in the first chapter are essential to understanding hypergeometric functions. The gamma function was introduced into mathematics by Euler when he solved the problem of extending the factorial function to all real or complex numbers. He could not have foreseen the extent of its importance in mathematics. There are extensions of gamma and beta functions that are also very important. The text contains a short treatment of Gauss and Jacobi sums, which are finite field analogs of gamma and beta functions. Gauss sums were encountered by Gauss in his work on the constructibility of regular polygons where they arose as "Lagrange resolvents," a concept used by Lagrange to study algebraic equations. Gauss understood the tremendous value of these sums for number theory. We discuss the derivation of Fermat's theorem on primes of the form $4n + 1$ from a formula connecting Gauss and Jacobi sums, which is analogous to Euler's famous formula relating beta integrals with gamma functions.

There are also multidimensional gamma and beta integrals. The first of these was introduced by Dirichlet, though it is really an iterated version of the one-dimensional integral. Genuine multidimensional gamma and beta functions were introduced in the 1930s, by both statisticians and number theorists. In the early 1940s, Atle Selberg found a very important multidimensional beta integral in the course of research in entire functions. However, owing to the Second World War and the fact that the first statement and also the proof appeared in journals that were not widely circulated, knowledge of this integral before the 1980s was restricted to a few people around the world. We present two different evaluations of Selberg's integral as well as some of its uses.

In addition to the above mentioned extensions, there are q-extensions of the gamma function and beta integrals that are very fundamental because they lead to basic hypergeometric functions and series. These are series Σc_n where c_{n+1}/c_n is a rational function of q^n for a fixed parameter q. Here the sum may run over all integers, instead of only nonnegative ones. One important example is the theta function $\sum_{-\infty}^{\infty} q^{n^2} x^n$. This and other similar series were used by Gauss and Jacobi to study elliptic and elliptic modular functions. Series of this sort are very useful in many areas of combinatorial analysis, a fact already glimpsed by Euler and Legendre, and they also arise in some branches of physics. For example, the work of the physicist R. J. Baxter on the Yang–Baxter equation led a group in St. Petersburgh to the notion of a quantum group. Independently, M. Jimbo in Japan was led by a study of Baxter's work to a related structure.

Many basic hypergeometric series (or q-hypergeometric series), both polynomials and infinite series, can be studied using Hopf algebras, which make up quantum groups. Unfortunately, we could not include this very important new approach to basic series. It was also not possible to include results on the multidimensional $U(n)$ generalizations of theorems on basic series, which have been studied extensively in recent years. For some of this work, the reader may refer to Milne [1988] and Milne and Lilly [1995]. We briefly discuss the q-gamma function and some important q-beta integrals; we show that series and products that arise in this theory have applications in number theory, combinatorics, and partition theory. We highlight the method of partition analysis.

P. A. MacMahon, who developed this powerful technique, devoted several chapters to it in his monumental *Combinatory Analysis*, but its significance was not realized until recently.

The theory of special functions with its numerous beautiful formulas is very well suited to an algorithmic approach to mathematics. In the nineteenth century, it was the ideal of Eisenstein and Kronecker to express and develop mathematical results by means of formulas. Before them, this attitude was common and best exemplified in the works of Euler, Jacobi, and sometimes Gauss. In the twentieth century, mathematics moved from this approach toward a more abstract and existential method. In fact, agreeing with Hardy that Ramanujan came 100 years too late, Littlewood once wrote that "the great day of formulae seem to be over" (see Littlewood [1986, p. 95]). However, with the advent of computers and the consequent reemergence of computational mathematics, formulas are now once again playing a larger role in mathematics. We present this book against this background, pointing out that beautiful, interesting, and important formulas have been discovered since Ramanujan's time. These formulas are proving fertile and fruitful; we suggest that the day of formulas may be experiencing a new dawn. Finally, we hope that the reader finds as much pleasure studying the formulas in this book as we have found in explaining them.

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