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The Gamma and Beta Functions

Euler discovered the gamma function, $\Gamma(x)$, when he extended the domain of the factorial function. Thus $\Gamma(x)$ is a meromorphic function equal to $(x-1)!$ when x is a positive integer. The gamma function has several representations, but the two most important, found by Euler, represent it as an infinite integral and as a limit of a finite product. We take the second as the definition.

Instead of viewing the beta function as a function, it is more illuminating to think of it as a class of integrals – integrals that can be evaluated in terms of gamma functions. We therefore often refer to beta functions as beta integrals.

In this chapter, we develop some elementary properties of the beta and gamma functions. We give more than one proof for some results. Often, one proof generalizes and others do not. We briefly discuss the finite field analogs of the gamma and beta functions. These are called Gauss and Jacobi sums and are important in number theory. We show how they can be used to prove Fermat's theorem that a prime of the form $4n+1$ is expressible as a sum of two squares. We also treat a simple multidimensional extension of a beta integral, due to Dirichlet, from which the volume of an n-dimensional ellipsoid can be deduced.

We present an elementary derivation of Stirling's asymptotic formula for $n!$ but give a complex analytic proof of Euler's beautiful reflection formula. However, two real analytic proofs due to Dedekind and Herglotz are included in the exercises. The reflection formula serves to connect the gamma function with the trigonometric functions. The gamma function has simple poles at zero and at the negative integers, whereas $\csc \pi x$ has poles at all the integers. The partial fraction expansions of the logarithmic derivatives of $\Gamma(x)$ motivate us to consider the Hurwitz and Riemann zeta functions. The latter function is of fundamental importance in the theory of distribution of primes. We have included a short discussion of the functional equation satisfied by the Riemann zeta function since it involves the gamma function.

In this chapter we also present Kummer's proof of his result on the Fourier expansion of $\log \Gamma(x)$. This formula is useful in number theory. The proof given uses Dirichlet's integral representations of $\log \Gamma(x)$ and its derivative. Thus, we have included these results of Dirichlet and the related theorems of Gauss.

1.1 The Gamma and Beta Integrals and Functions

The problem of finding a function of a continuous variable x that equals $n!$ when $x = n$, an integer, was investigated by Euler in the late 1720s. This problem was apparently suggested by Daniel Bernoulli and Goldbach. Its solution is contained in Euler's letter of October 13, 1729, to Goldbach. See Fuss [1843, pp. 1–18]. To arrive at Euler's generalization of the factorial, suppose that $x \geq 0$ and $n \geq 0$ are integers. Write

$$x! = \frac{(x+n)!}{(x+1)_n}, \tag{1.1.1}$$

where $(a)_n$ denotes the shifted factorial defined by

$$(a)_n = a(a+1)\cdots(a+n-1) \quad \text{for } n > 0,\ (a)_0 = 1, \tag{1.1.2}$$

and a is any real or complex number. Rewrite (1.1.1) as

$$x! = \frac{n!(n+1)_x}{(x+1)_n} = \frac{n!n^x}{(x+1)_n} \cdot \frac{(n+1)_x}{n^x}.$$

Since

$$\lim_{n\to\infty} \frac{(n+1)_x}{n^x} = 1,$$

we conclude that

$$x! = \lim_{n\to\infty} \frac{n!n^x}{(x+1)_n}. \tag{1.1.3}$$

Observe that, as long as x is a complex number not equal to a negative integer, the limit in (1.1.3) exists, for

$$\frac{n!n^x}{(x+1)_n} = \left(\frac{n}{n+1}\right)^x \prod_{j=1}^{n} \left(1+\frac{x}{j}\right)^{-1} \left(1+\frac{1}{j}\right)^x$$

and

$$\left(1+\frac{x}{j}\right)^{-1} \left(1+\frac{1}{j}\right)^x = 1 + \frac{x(x-1)}{2j^2} + O\left(\frac{1}{j^3}\right).$$

Therefore, the infinite product

$$\prod_{j=1}^{\infty}\left(1+\frac{x}{j}\right)^{-1}\left(1+\frac{1}{j}\right)^{x}$$

converges and the limit (1.1.3) exists. (Readers who are unfamiliar with infinite products should consult Appendix A.) Thus we have a function

$$\Pi(x)=\lim_{k\to\infty}\frac{k!k^{x}}{(x+1)_{k}} \tag{1.1.4}$$

defined for all complex $x \neq -1, -2, -3, \ldots$ and $\Pi(n) = n!$.

Definition 1.1.1 *For all complex numbers $x \neq 0, -1, -2, \ldots$, the gamma function $\Gamma(x)$ is defined by*

$$\Gamma(x)=\lim_{k\to\infty}\frac{k!k^{x-1}}{(x)_{k}}. \tag{1.1.5}$$

An immediate consequence of Definition 1.1.1 is

$$\Gamma(x+1)=x\Gamma(x). \tag{1.1.6}$$

Also,

$$\Gamma(n+1)=n! \tag{1.1.7}$$

follows immediately from the above argument or from iteration of (1.1.6) and use of

$$\Gamma(1)=1. \tag{1.1.8}$$

From (1.1.5) it follows that the gamma function has poles at zero and the negative integers, but $1/\Gamma(x)$ is an entire function with zeros at these points. Every entire function has a product representation; the product representation of $1/\Gamma(x)$ is particularly nice.

Theorem 1.1.2

$$\frac{1}{\Gamma(x)}=xe^{\gamma x}\prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{n}\right)e^{-x/n}\right\}, \tag{1.1.9}$$

where γ is Euler's constant given by

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right). \tag{1.1.10}$$

Proof.

$$\begin{aligned}\frac{1}{\Gamma(x)} &= \lim_{n\to\infty} \frac{x(x+1)\cdots(x+n-1)}{n!n^{x-1}} \\ &= \lim_{n\to\infty} x\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\cdots\left(1+\frac{x}{n}\right)e^{-x\log n} \\ &= \lim_{n\to\infty} xe^{x(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)}\prod_{k=1}^{n}\left\{\left(1+\frac{x}{k}\right)e^{-x/k}\right\} \\ &= xe^{\gamma x}\prod_{n=1}^{\infty}\left\{\left(1+\frac{x}{n}\right)e^{-x/n}\right\}.\end{aligned}$$

The infinite product in (1.1.9) exists because

$$\left(1+\frac{x}{n}\right)e^{-x/n} = \left(1+\frac{x}{n}\right)\left(1-\frac{x}{n}+\frac{x^2}{2n^2}\cdots\right) = 1-\frac{x^2}{2n^2}+O\left(\frac{1}{n^3}\right),$$

and the factor $e^{-x/n}$ was introduced to make this possible. The limit in (1.1.10) exists because the other limits exist, or its existence can be shown directly. One way to do this is to show that the difference between adjacent expressions under the limit sign decay in a way similar to $1/n^2$. ■

One may take (1.1.9) as a definition of $\Gamma(x)$ as Weierstrass did, though the formula had been found earlier by Schlömilch and Newman. See Nielsen [1906, p. 10].

Over seventy years before Euler, Wallis [1656] attempted to compute the integral $\int_0^1 \sqrt{1-x^2}dx = \frac{1}{2}\int_{-1}^{+1}(1-x)^{1/2}(1+x)^{1/2}dx$. Since this integral gives the area of a quarter circle, Wallis's aim was to obtain an expression for π. The only integral he could actually evaluate was $\int_0^1 x^p(1-x)^q dx$, where p and q are integers or $q=0$ and p is rational. He used the value of this integral and some audacious guesswork to suggest that

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2}dx = \frac{1}{4}\lim_{n\to\infty}\left[\frac{2\cdot 4\cdot 6\cdots 2n}{1\cdot 3\cdot 5\cdots(2n-1)}\cdot\frac{1}{\sqrt{n}}\right]^2 = \Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right). \tag{1.1.11}$$

Of course, he did not write it as a limit or use the gamma function. Still, this result may have led Euler to consider the relation between the gamma function and integrals of the form $\int_0^1 x^p(1-x)^q dx$ where p and q are not necessarily integers.

Definition 1.1.3 *The beta integral is defined for* Re $x>0$, Re $y>0$ *by*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{1.1.12}$$

One may also speak of the beta function $B(x, y)$, which is obtained from the integral by analytic continuation.

The integral (1.1.12) is symmetric in x and y as may be seen by the change of variables $u = 1 - t$.

Theorem 1.1.4

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1.13}$$

Remark 1.1.1 The essential idea of the proof given below goes back to Euler [1730, 1739] and consists of first setting up a functional relation for the beta function and then iterating the relation. An integral representation for $\Gamma(x)$ is obtained as a byproduct. The functional equation technique is useful for evaluating certain integrals and infinite series; we shall see some of its power in subsequent chapters.

Proof. The functional relation we need is

$$B(x, y) = \frac{x+y}{y} B(x, y+1). \tag{1.1.14}$$

First note that for Re $x > 0$ and Re $y > 0$,

$$\begin{aligned} B(x, y+1) &= \int_0^1 t^{x-1}(1-t)(1-t)^{y-1} dt \\ &= B(x, y) - B(x+1, y). \end{aligned} \tag{1.1.15}$$

However, integration by parts gives

$$\begin{aligned} B(x, y+1) &= \left[\frac{1}{x} t^x (1-t)^y\right]_0^1 + \frac{y}{x} \int_0^1 t^x (1-t)^{y-1} dt \\ &= \frac{y}{x} B(x+1, y). \end{aligned} \tag{1.1.16}$$

Combine (1.1.15) and (1.1.16) to get the functional relation (1.1.14). Other proofs of (1.1.14) are given in problems at the end of this chapter. Now iterate (1.1.14) to obtain

$$B(x, y) = \frac{(x+y)(x+y+1)}{y(y+1)} B(x, y+2) = \cdots = \frac{(x+y)_n}{(y)_n} B(x, y+n).$$

Rewrite this relation as

$$\begin{aligned} B(x, y) &= \frac{(x+y)_n}{n!} \frac{n!}{(y)_n} \int_0^n \left(\frac{t}{n}\right)^{x-1} \left(1 - \frac{t}{n}\right)^{y+n-1} \frac{dt}{n} \\ &= \frac{(x+y)_n}{n! n^{x+y-1}} \frac{n! n^{y-1}}{(y)_n} \int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^{n+y-1} dt. \end{aligned}$$

As $n \to \infty$, the integral tends to $\int_0^\infty t^{x-1}e^{-t}dt$. This may be justified by the Lebesgue dominated convergence theorem. Thus

$$B(x, y) = \frac{\Gamma(y)}{\Gamma(x+y)} \int_0^\infty t^{x-1}e^{-t}dt. \tag{1.1.17}$$

Set $y = 1$ in (1.1.12) and (1.1.17) to get

$$\frac{1}{x} = \int_0^1 t^{x-1}dt = B(x, 1) = \frac{\Gamma(1)}{\Gamma(x+1)} \int_0^\infty t^{x-1}e^{-t}dt.$$

Then (1.1.6) and (1.1.8) imply that $\int_0^\infty t^{x-1}e^{-t}dt = \Gamma(x)$ for $\operatorname{Re} x > 0$. Now use this in (1.1.17) to prove the theorem for $\operatorname{Re} x > 0$ and $\operatorname{Re} y > 0$. The analytic continuation is immediate from the value of this integral, since the gamma function can be analytically continued. ■

Remark 1.1.2 Euler's argument in [1739] for (1.1.13) used a recurrence relation in x rather than in y. This leads to divergent infinite products and an integral that is zero. He took two such integrals, with y and $y = m$, divided them, and argued that the resulting "vanishing" integrals were the same. These canceled each other when he took the quotient of the two integrals with y and $y = m$. The result was an infinite product that converges and gives the correct answer. Euler's extraordinary intuition guided him to correct results, even when his arguments were as bold as this one.

Earlier, in 1730, Euler had evaluated (1.1.13) by a different method. He expanded $(1-t)^{y-1}$ in a series and integrated term by term. When $y = n + 1$, he stated the value of this sum in product form.

An important consequence of the proof is the following corollary:

Corollary 1.1.5 *For* $\operatorname{Re} x > 0$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt. \tag{1.1.18}$$

The above integral for $\Gamma(x)$ is sometimes called the Eulerian integral of the second kind. It is often taken as the definition of $\Gamma(x)$ for $\operatorname{Re} x > 0$. The Eulerian integral of the first kind is (1.1.12). Legendre introduced this notation. Legendre's $\Gamma(x)$ is preferred over Gauss's function $\Pi(x)$ given by (1.1.4), because Theorem 1.1.4 does not have as nice a form in terms of $\Pi(x)$. For another reason, see Section 1.10.

The gamma function has poles at zero and at the negative integers. It is easy to use the integral representation (1.1.18) to explicitly represent the poles and the analytic continuation of $\Gamma(x)$:

$$\begin{aligned}\Gamma(x) &= \int_0^1 t^{x-1}e^{-t}dt + \int_1^\infty t^{x-1}e^{-t}dt \\ &= \sum_{n=0}^\infty \frac{(-1)^n}{(n+x)n!} + \int_1^\infty t^{x-1}e^{-t}dt. \end{aligned} \tag{1.1.19}$$

The second function on the right-hand side is an entire function, and the first shows that the poles are as claimed, with $(-1)^n/n!$ being the residue at $x = -n, n = 0, 1, \dots$.

The beta integral has several useful forms that can be obtained by a change of variables. For example, set $t = s/(s+1)$ in (1.1.12) to obtain the beta integral on a half line,

$$\int_0^\infty \frac{s^{x-1}}{(1+s)^{x+y}}ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1.20}$$

Then again, take $t = \sin^2\theta$ to get

$$\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}. \tag{1.1.21}$$

Put $x = y = 1/2$. The result is

$$\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{2\Gamma(1)} = \frac{\pi}{2},$$

or

$$\Gamma(1/2) = \sqrt{\pi}. \tag{1.1.22}$$

Since this implies $[\Gamma(\frac{3}{2})]^2 = \pi/4$, we have a proof of Wallis's formula (1.1.11). We also have the value of the normal integral

$$\int_{-\infty}^\infty e^{-x^2}dx = 2\int_0^\infty e^{-x^2}dx = \int_0^\infty t^{-1/2}e^{-t}dt = \Gamma(1/2) = \sqrt{\pi}. \tag{1.1.23}$$

Finally, the substitution $t = (u-a)/(b-a)$ in (1.1.12) gives

$$\int_a^b (b-u)^{x-1}(u-a)^{y-1}du = (b-a)^{x+y-1}B(x,y) = (b-a)^{x+y-1}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1.24}$$

The special case $a = -1, b = 1$ is worth noting as it is often used:

$$\int_{-1}^1 (1+t)^{x-1}(1-t)^{y-1}dt = 2^{x+y-1}\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.1.25}$$

A useful representation of the analytically continued beta function is

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \frac{(x+y)}{xy} \prod_{n=1}^{\infty} \frac{\left(1+\frac{x+y}{n}\right)}{\left(1+\frac{x}{n}\right)\left(1+\frac{y}{n}\right)}. \tag{1.1.26}$$

This follows immediately from Theorem 1.1.2. Observe that $B(x, y)$ has poles at x and y equal to zero or negative integers, and it is analytic elsewhere.

As mentioned before, the integral formula for $\Gamma(x)$ is often taken as the definition of the gamma function. One reason is that the gamma function very frequently appears in this form. Moreover, the basic properties of the function can be developed easily from the integral. We have the powerful tools of integration by parts and change of variables that can be applied to integrals. As an example, we give another derivation of Theorem 1.1.4. This proof is also important because it can be applied to obtain the finite field analog of Theorem 1.1.4. In that situation one works with a finite sum instead of an integral.

Poisson [1823] and independently Jacobi [1834] had the idea of starting with an appropriate double integral and evaluating it in two different ways. Thus, since the integrals involved are absolutely convergent,

$$\int_0^\infty \int_0^\infty t^{x-1} s^{y-1} e^{-(s+t)} ds dt = \int_0^\infty t^{x-1} e^{-t} dt \int_0^\infty s^{y-1} e^{-s} ds = \Gamma(x)\Gamma(y).$$

Apply the change of variables $s = uv$ and $t = u(1-v)$ to the double integral, and observe that $0 < u < \infty$ and $0 < v < 1$ when $0 < s, t < \infty$. This change of variables is suggested by first setting $s + t = u$. Computation of the Jacobian gives $dsdt = ududv$ and the double integral is transformed to

$$\int_0^\infty e^{-u} u^{x+y-1} du \int_0^1 v^{x-1}(1-v)^{y-1} dv = \Gamma(x+y)B(x, y).$$

A comparison of two evaluations of the double integral gives the necessary result. This is Jacobi's proof. Poisson's proof is similar except that he applies the change of variables $t = r$ and $s = ur$ to the double integral. In this case the beta integral obtained is on the interval $(0, \infty)$ as in (1.1.20). See Exercise 1.

To complete this section we show how the limit formula for $\Gamma(x)$ can be derived from an integral representation of $\Gamma(x)$. We first prove that when n is an integer ≥ 0 and $\operatorname{Re} x > 0$,

$$\int_0^1 t^{x-1}(1-t)^n dt = \frac{n!}{x(x+1)\cdots(x+n)}. \tag{1.1.27}$$

This is actually a special case of Theorem 1.1.4 but we give a direct proof by induction, in order to avoid circularity in reasoning. Clearly (1.1.27) is true for

$n = 0$, and

$$\int_0^1 t^{x-1}(1-t)^{n+1}dt = \int_0^1 t^{x-1}(1-t)(1-t)^n dt$$
$$= \frac{n!}{(x)_{n+1}} - \frac{n!}{(x+1)_{n+1}}$$
$$= \frac{(n+1)!}{(x)_{n+2}}.$$

This proves (1.1.27) inductively. Now set $t = u/n$ and let $n \to \infty$. By the Lebesgue dominated convergence theorem it follows that

$$\int_0^\infty t^{x-1}e^{-t}dt = \lim_{n\to\infty} \frac{n!n^{x-1}}{(x)_n} \quad \text{for } \operatorname{Re} x > 0.$$

Thus, if we begin with the integral definition for $\Gamma(x)$ then the above formula can be used to extend it to other values of x (i.e., those not equal to $0, -1, -2, \ldots$).

Remark 1.1.3 It is traditional to call the integral (1.1.12) the beta function. A better terminology might call this Euler's first beta integral and call (1.1.20) the second beta integral. We call the integral in Exercise 13 Cauchy's beta integral. We shall study other beta integrals in later chapters, but the common form of these three is $\int_C [\ell_1(t)]^p [\ell_2(t)]^q dt$, where $\ell_1(t)$ and $\ell_2(t)$ are linear functions of t, and C is an appropriate curve. For Euler's first beta integral, the curve consists of a line segment connecting the two zeros; for the second beta integral, it is a half line joining one zero with infinity such that the other zero is not on this line; and for Cauchy's beta integral, it is a line with zeros on opposite sides. See Whittaker and Watson [1940, §12.43] for some examples of beta integrals that contain curves of integration different from those mentioned above. An important one is given in Exercise 54.

1.2 The Euler Reflection Formula

Among the many beautiful formulas involving the gamma function, the Euler reflection formula is particularly significant, as it connects the gamma function with the sine function. In this section, we derive this formula and briefly describe how product and partial fraction expansions for the trigonometric functions can be obtained from it. Euler's formula given in Theorem 1.2.1 shows that, in a sense, the function $1/\Gamma(x)$ is half of the sine function.

Theorem 1.2.1 *Euler's reflection formula:*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{1.2.1}$$

Remark The proof given here uses contour integration. Since the gamma function is a real variable function in the sense that many of its important characterizations occur within that theory, three real variable proofs are outlined in the Exercises. See Exercises 15, 16, and 26–27.

Since we shall show how some of the theory of trigonometric functions can be derived from (1.2.1), we now state that $\sin x$ is here defined by the series

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

The cosine function is defined similarly. It is easy to show from this definition that sine and cosine have period 2π and that $e^{\pi i} = -1$. See Rudin [1976, pp. 182–184].

Proof. Set $y = 1 - x$, $0 < x < 1$ in (1.1.20) to obtain

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{t^{x-1}}{1+t} dt. \tag{1.2.2}$$

To compute the integral in (1.2.2), consider the integral

$$\int_C \frac{z^{x-1}}{1-z} dz,$$

where C consists of two circles about the origin of radii R and ϵ respectively, which are joined along the negative real axis from $-R$ to $-\epsilon$. Move along the outer circle in the counterclockwise direction, and along the inner circle in the clockwise direction. By the residue theorem

$$\int_C \frac{z^{x-1}}{1-z} dz = -2\pi i, \tag{1.2.3}$$

when z^{x-1} has its principal value. Thus

$$-2\pi i = \int_{-\pi}^{\pi} \frac{iR^x e^{ix\theta}}{1 - Re^{i\theta}} d\theta + \int_R^\epsilon \frac{t^{x-1} e^{ix\pi}}{1+t} dt + \int_\pi^{-\pi} \frac{i\epsilon^x e^{ix\theta}}{1 - \epsilon e^{i\theta}} d\theta + \int_\epsilon^R \frac{t^{x-1} e^{-ix\pi}}{1+t} dt.$$

Let $R \to \infty$ and $\epsilon \to 0$ so that the first and third integrals tend to zero and the second and fourth combine to give (1.2.1) for $0 < x < 1$. The full result follows by analytic continuation. One could also argue as follows: Equality of (1.2.1) for $0 < x < 1$ implies equality in $0 < \operatorname{Re} x < 1$ by analyticity; for $\operatorname{Re} x = 0, x \neq 0$ by continuity; and then for x shifted by integers using $\Gamma(x+1) = x\Gamma(x)$ and $\sin(x + \pi) = -\sin x$. ∎

The next theorem is an immediate consequence of Theorem 1.2.1.

Theorem 1.2.2

$$\sin \pi x = \pi x \prod_{n=1}^\infty \left(1 - \frac{x^2}{n^2}\right), \tag{1.2.4}$$