COHOMOLOGY OF VECTOR BUNDLES AND SYZYGIES

The central theme of this book is an exposition of the geometric technique of calculating syzygies. It is written from the point of view of commutative algebra; without assuming any knowledge of representation theory, the calculation of syzygies of determinantal varieties is explained. The starting point is a definition of Schur functors, and these are discussed from both an algebraic and a geometric point of view. Then a chapter on various versions of Bott's theorem leads to a careful explanation of the technique itself, based on a description of the direct image of a Koszul complex. Applications to determinantal varieties follow. There are also chapters on applications of the technique to rank varieties for symmetric and skew symmetric tensors of arbitrary degree, closures of conjugacy classes of nilpotent matrices, discriminants, and resultants. Numerous exercises are included to give the reader insight into how to apply this important method.

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Preface

This book is devoted to the geometric technique of calculating syzygies. This technique originated with George Kempf and was first used successfully by Alain Lascoux for calculating syzygies of determinantal varieties. Since then it has been applied in studying the defining ideals of varieties with symmetries that play a central role in geometry: determinantal varieties, closures of nilpotent orbits, and discriminant and resultant varieties.

The character of the method makes it comparable to the symbolic method in classical invariant theory. It works in only a limited number of cases, but when it does, it gives a complete answer to the problem of calculating syzygies, and this answer is hard to get by other means.

Even though the basic idea is more than 20 years old, this is the first book treating the geometric technique in detail. This happens because authors using the geometric method have usually been interested more in the special cases they studied than in the method itself, and they have used only the aspects of the method they needed. Therefore the basic theorems from chapter 5 stem from the efforts of several mathematicians.

The possibilities offered by the geometric method are not exhausted by the examples treated in the book. The method can be fruitfully applied to any representation of a linearly reductive group with finitely many orbits and with actions such that the orbits can be described explicitly. The varieties treated in chapters 7 and 9 show that the scope of the method is not limited to such actions.

The book is written from the point of view of a commutative algebraist. We develop the rudiments of representation theory of general linear group in some detail in chapters 2–4. At the same time we assume some knowledge of commutative algebra and algebraic geometry, including sheaves and their cohomology—for example, the notions covered in chapters II and III of the book [H1] of Hartshorne. The notions of Cohen–Macaulay and Gorenstein rings and rational singularities are also used, and their definitions are briefly recalled in chapter 1.

Some parts of the book demand more advanced knowledge. One statement in chapter 5 is an application of Grothendieck duality, whose statement is briefly recalled in chapter 1. We advise less experienced readers to just accept Theorem (5.1.4) and look first at its applications.

In some sections of chapters 5 and 8 we assume familiarity with highest weight theory and some facts on linear algebraic groups. At the same time, in exercises we use the geometric technique to develop some of that theory for the classical groups. Still, the reader not familiar with these notions should be able to understand all the remaining chapters.

Let us describe briefly the contents of the book. The first chapter discusses preliminaries. We recall elementary notions from multilinear algebra and combinatorics. The section on commutative and homological algebra covers briefly the definitions of depth, Koszul complexes, the Auslander–Buchsbaum–Serre theorem, and Cohen–Macaulay and Gorenstein rings. There is also a section on de Jong's algorithm for explicit calculation of normalization, on the exactness criterion of Buchsbaum and Eisenbud, and on Grothendieck duality. The final section is devoted to a brief review of the notion of the determinant of a complex.

Chapters 2, 3, and 4 are devoted to developing the representation theory of general linear groups and to the proof of Bott's theorem on the cohomology of line bundles on homogeneous spaces. This provides the basic tools for the calculations to be performed in later chapters.

Our approach is based on Schur and Weyl functors, introduced in the first section of chapter 2. They are defined by generators and relations. The relation to highest weight theory and Schur–Young theory is discussed. This is followed by a discussion of Cauchy formulas, the Littlewood–Richardson rule, and plethysm. The final section of chapter 2 discusses Schur complexes.

In chapter 3 we relate Schur functors to geometry by realizing them as multihomogeneous components of homogeneous coordinate rings of flag varieties. This is followed by a proof of the Cauchy formula based on restriction of the straightening from the Grassmannian to its affine open subset, and by a section on tangent bundles of Grassmannians and flag varieties.

In chapter 4 we prove Bott's theorem on cohomology of line bundles on flag varieties. We follow the approach of Demazure. In the last section we formulate the theorem for arbitrary reductive groups and give explicit interpretations for classical groups.

Chapter 5 is devoted to the description of terms and properties of the direct images of Koszul complexes. We study the basic setup, i.e., the diagram

$$\begin{array}{ccc} Z & \subset & X \times V \\ \downarrow q' & & \downarrow q \\ Y & \subset & X \end{array}$$

where X is an affine space, V is a nonsingular projective variety, Z is a total space of a vector subbundle $\mathcal{S}$ of the trivial bundle $X \times V$, and $Y = q(Z)$. The variety Z can be described as the vanishing set of a cosection $p^*(\xi) \to \mathcal{O}_{X\times V}$, and it is a locally complete intersection. Here ξ is the dual of the factor $(X \times V)/\mathcal{S}$. The original idea of Kempf was that the study of a direct image of the resulting Koszul complex can be used to prove results about the defining equations and syzygies of the subvariety Y. We give the basic properties of these direct images in the general case and in important special cases, for example when Z is a desingularization of Y. We also treat the more general case of twisted Koszul complexes.

The remaining chapters are devoted to examples and applications. In each of these chapters a different aspect of the method is illustrated.

In chapter 6 we study the case of determinantal varieties. Here we show how to handle basic calculations in simple cases when the bundle ξ is a tensor product of tautological bundles. Apart from the proof of Lascoux's theorem and the calculation of syzygies of determinantal ideals for symmetric and skew symmetric matrices, we also give results on the equivariant modules supported in determinantal varieties.

Chapter 7 is devoted to the rank varieties for tensors of degree higher than two. This illustrates that the method can be applied in cases when the variety X or even Y does not have finitely many orbits with respect to some action of the reductive group.

In chapter 8 the study of nilpotent orbit closures allows us to understand how to handle the situation when the cohomology groups needed to get the syzygies cannot be calculated directly, but still partial results can be recovered by estimating the terms in a spectral sequence associated to filtrations on a basic bundle ξ. We also prove the Hinich–Panyushev theorem on rational singularities of normalizations of nilpotent orbits for a general simple group.

Chapter 9 illustrates the use of twisted modules supported in resultant and discriminant varieties, which allow one to get natural determinantal expressions for the defining equation.

Each chapter is followed by exercises. They should allow readers to learn how to apply the geometric method on their own. At the same time they illustrate further applications of the method. In particular the exercises to chapter 6 deal with the analogues of the determinantal varieties for the symplectic and orthogonal groups. In exercises to chapter 7 we give some calculations of minimal resolutions of Plücker ideals.

The book can be read on several levels. For the reader who is not familiar with representation theory and/or derived categories and Grothendieck duality, we suggest first reading chapters 2 through 4. Then one can proceed with the proof of the statement of Theorem 5.1.2 given in section 5.2. At that point most of the following chapters (with the exception of section 8.3) can be understood using only that statement.

In such a way the book could be used as the basis of an advanced course in commutative algebra or algebraic geometry. It could also serve as the basis of a seminar. A lot of general notions and theories can be nicely illustrated in the special cases treated in later chapters by the methods given in the book.

A reader familiar with representation theory can just skim through chapters 2 through 4 to get familiar with the notation, and then proceed straight to chapter 5 and study the applications.

I am indebted to many people who introduced me to the subject, especially to David Buchsbaum, Corrado De Concini, Jack Eagon, David Eisenbud, Tadeusz Józefiak, Piotr Pragacz, and Joel Roberts. I also benefited from conversations on some aspects of the material with Kaan Akin, Giandomenico Boffi, Michel Brion, Bram Broer, Andrzej Daszkiewicz, Steve Donkin, Toshizumi Fukui, Laura Galindo, Wilberd van der Kallen, Jacek Klimek, Hanspeter Kraft, Witold Kraśkiewicz, Alain Lascoux, Steve Lovett, Olga Porras, Claudio Procesi, Rafael Sanchez, Mark Shimozono, Alex Tchernev, and Andrei Zelevinsky. Throughout my work on the book I was partially supported by grants from the National Science Foundation.