1

Introductory Material

1.1. Multilinear Algebra and Combinatorics

1.1.1. Exterior, Divided, and Symmetric Powers; Multiplication and Diagonal Maps

Let $\mathbf{K}$ be a commutative ring, and let E be a free $\mathbf{K}$-module with a basis $\{e_1, \dots, e_n\}$.

We define *the r-th exterior power* $\bigwedge^r E$ *of* E to be the r-th tensor power $E^{\otimes r}$ of E divided by the submodule generated by the elements:

$$u_1 \otimes \dots \otimes u_r - (-1)^{\operatorname{sgn}\sigma} u_{\sigma(1)} \otimes \dots \otimes u_{\sigma(r)}$$

for all $\sigma \in \Sigma_r$, $u_1, \dots, u_r \in E$. We denote the coset of $u_1 \otimes \dots \otimes u_r$ by $u_1 \wedge \dots \wedge u_r$.

The following basic properties of exterior powers are proved in [L, chapter XIX, section 1].

(1.1.1) Proposition.

(a) *Let $\{e_1, \dots, e_n\}$ be an ordered basis of E. Then the elements $e_{i_1} \wedge \dots \wedge e_{i_r}$ for $1 \le i_1 < \dots < i_r \le n$ form a basis of $\bigwedge^r E$. In particular, $\bigwedge^r E$ is a free* $\mathbf{K}$*-module of dimension $\binom{n}{r}$.*

(b) *(Universality property of exterior powers) We have a functorial isomorphism*

$$\theta_M : \operatorname{Alt}^r(E^r, M) \to \operatorname{Hom}_{\mathbf{K}}\left(\bigwedge^r E, M\right)$$

where $\operatorname{Alt}^r(E^r, M)$ denotes the set of multilinear alternating maps from $E^{\times r}$ to M, given by the formula $\theta^r_M(f)(u_1 \wedge \dots \wedge u_r) = f(u_1, \dots, u_r)$.

(c) *We have natural isomorphisms*

$$\alpha^r : \bigwedge^r (E^*) \to \left(\bigwedge^r E\right)^*$$

sending the exterior product $l_1 \wedge \ldots \wedge l_r$ *to the linear function* l *on* $\bigwedge^e E$ *defined by the formula*

$$l(u_1 \wedge \ldots \wedge u_r) = \sum_{\sigma \in \Sigma^r} (-1)^{\operatorname{sgn} \sigma} l_{\sigma(1)}(u_1) \ldots l_{\sigma(r)}(u_r).$$

The r-th exterior power is an endofunctor on the category of free **K**-modules and linear maps. More precisely, for two free **K**-modules E, F and a linear map $\phi : E \to F$ we have a well-defined linear map

$$\bigwedge^r \phi : \bigwedge^r E \to \bigwedge^r F$$

defined by the formula $\bigwedge^r \phi(u_1 \wedge \ldots \wedge u_r) = \phi(u_1) \wedge \ldots \wedge \phi(u_r)$. Let us denote $m = \dim F$. Let $\{e_1, \ldots, e_n\}$ be a basis of E and let $\{f_1, \ldots, f_m\}$ be a basis of F. In these bases ϕ correspond to the $m \times n$ matrix $(\phi_{j,i})$ where

$$\phi(e_i) = \sum_{j=1}^{m} \phi_{j,i} f_j.$$

The map $\bigwedge^r \phi$ can be written in the corresponding bases of $\bigwedge^r E$, $\bigwedge^r F$ as follows:

$$\bigwedge^r \phi(e_{i_1} \wedge \ldots \wedge e_{i_r}) = \sum_{1 \le j_1 < \ldots < j_r \le m} M(j_1, \ldots, j_r | i_1, \ldots, i_r; \phi) f_{j_1} \wedge \ldots \wedge f_{j_r},$$

where $M(j_1, \ldots, j_r | i_1, \ldots, i_r; \phi)$ denotes the $r \times r$ minor of the matrix $(\phi_{j,i})$ corresponding to the rows $j_1, \ldots, j_r$ and columns $i_1, \ldots, i_r$.

The vector space

$$\bigwedge^\bullet(E) := \bigoplus_{r \ge 0} \bigwedge^r E$$

has a natural multiplication

$$m : \bigwedge^\bullet(E) \otimes \bigwedge^\bullet(E) \to \bigwedge^\bullet(E)$$

given by the formula

$$m(u_1 \wedge \ldots u_r \otimes v_1 \wedge \ldots \wedge v_s) = u_1 \wedge \ldots \wedge u_r \wedge v_1 \wedge \ldots \wedge v_s.$$

This gives $\bigwedge^\bullet(E)$ the structure of associative, graded commutative algebra (meaning that the commutative law reads $fg = (-1)^{\deg(f)\deg(g)} gf$). We call this algebra *the exterior algebra on* E. The algebra $\bigwedge^\bullet(E)$ has a unit $\eta : \mathbf{K} \to \bigwedge^\bullet(E)$.

The components of the multiplication map will be denoted by $m : \bigwedge^r E \otimes \bigwedge^s E \to \bigwedge^{r+s} E$.

The diagonal map $\Delta : E \to E \oplus E$ induces an algebra map

$$\Delta : \bigwedge^\bullet(E) \to \bigwedge^\bullet(E \oplus E) \cong \bigwedge^\bullet(E) \otimes \bigwedge^\bullet(E)$$

which we will call *the diagonal* (or *comultiplication*) *map*.

The components of Δ will be denoted by $\Delta : \bigwedge^{r+s} E \to \bigwedge^r E \otimes \bigwedge^s E$. In terms of elements we have

$$\begin{aligned}\Delta\ (u_1 \wedge \ldots \wedge u_{r+s}) \\ = \sum_{\sigma \in \Sigma^{r,s}_{r+s}} (-1)^{\mathrm{sgn}\ \sigma} u_{\sigma(1)} \wedge \ldots \wedge u_{\sigma(r)} \otimes u_{\sigma(r+1)} \wedge \ldots \wedge u_{\sigma(r+s)}\end{aligned}$$

where $\Sigma^{r,s}_{r+s} = \{\sigma \in \Sigma_{r+s} \mid \sigma(1) < \ldots < \sigma(r);\ \ \sigma(r+1) < \ldots < \sigma(r+s)\}$.

Finally we have the counit map

$$\epsilon : \bigwedge^\bullet(E) \to \mathbf{K},$$

defined to be zero on all spaces $\bigwedge^r E$ for $r > 0$, and satisfying $\epsilon\eta(1) = 1$.

The following proposition is an elementary calculation.

(1.1.2) Proposition.

(a) The maps $m, \Delta, \epsilon, \eta$ define on $\bigwedge^\bullet(E)$ the structure of commutative, cocommutative bialgebra.

(b) The map $\alpha : \bigwedge^\bullet(E^*) \to (\bigwedge^\bullet E)^*$ defined in (1.1.1) (c) is an isomorphism of bialgebras.

Part (b) of the proposition means that the dual map to the multiplication map m on $\bigwedge^\bullet(E)$ is the diagonal map Δ on $\bigwedge^\bullet(E^*)$ and vice versa.

We define *the r-th symmetric power $S_r E$ of E* to be the r-th tensor power $E^{\otimes r}$ of E divided by the submodule generated by the elements

$$u_1 \otimes \ldots \otimes u_r - u_{\sigma(1)} \otimes \ldots \otimes u_{\sigma(r)}$$

for all $\sigma \in \Sigma_r$, $u_1, \ldots, u_r \in E$. We denote the coset of $u_1 \otimes \ldots \otimes u_r$ by $u_1 \ldots u_r$.

The following basic properties of symmetric powers are proved in [L, chapter XVI, section 8].

(1.1.3) Proposition.

(a) Let $\{e_1, \ldots, e_n\}$ be an ordered basis of E. Then the elements $e_1^{i_1} \ldots e_n^{i_n}$ for $i_1 + \ldots + i_n = r$ form a basis of $S_r E$. In particular $S_r E$ is a free **K***-module of dimension $\binom{n+r-1}{r}$.*

(b) (Universality property of symmetric powers) We have a functorial isomorphism

$$\theta_M : \mathrm{Sym}^r(E^r, M) \to \mathrm{Hom}_{\mathbf{K}}(S_r E, M)$$

where $\mathrm{Sym}^r(E^r, M)$ denotes the set of multilinear symmetric maps from $E^{\times r}$ to M, given by the formula $\theta_M^r(f)(u_1 \ldots u_r) = f(u_1, \ldots, u_r)$.

The r-th symmetric power is an endofunctor on the category of free **K**-modules and linear maps. More precisely, for two free **K**-modules E, F and a linear map $\phi : E \to F$ we have a well-defined linear map

$$S_r\phi : S_r E \to S_r F$$

defined by the formula $S_r\phi(u_1 \ldots u_r) = \phi(u_1) \ldots \phi(u_r)$. Let us denote $m = \dim F$. Let $\{e_1, \ldots, e_n\}$ be a basis of E, and let $\{f_1, \ldots, f_m\}$ be a basis of F. In these bases ϕ correspond to the $m \times n$ matrix $(\phi_{j,i})$ where

$$\phi(e_i) = \sum_{j=1}^{m} \phi_{j,i} f_j.$$

The map $S_r\phi$ can be written in the corresponding bases of $S_r E$, $S_r F$ as follows:

$$S_r\phi(e_{i_1} \ldots e_{i_r}) = \sum_{1 \le j_1 < \ldots < j_r \le m} P(j_1, \ldots, j_r | \, i_1, \ldots, i_r; \phi) f_{j_1} \ldots f_{j_r},$$

where $P(j_1, \ldots, j_r | \, i_1, \ldots, i_r; \phi)$ denotes the permanent of the $r \times r$ submatrix of the matrix $(\phi_{j,i})$ corresponding to the (possibly repeated) rows $j_1, \ldots, j_r$ and (possibly repeated) columns $i_1, \ldots, i_r$. More precisely, if the columns $(i_1, \ldots, i_r)$ with repetitions are written as $i_1^{b_1}, \ldots, i_s^{b_s}$ with $b_1 + \ldots + b_s = r$, we have

$$P(j_1, \ldots, j_r | \, i_1^{b_1}, \ldots, i_s^{b_s}) = \sum_{\sigma \in \Sigma_r / (\Sigma_{b_1} \times \Sigma_{b_s})} \phi(j_1, i_{\sigma(1)}) \ldots \phi(j_r, i_{\sigma(r)}).$$

where $\Sigma_{b_1} \times \ldots \times \Sigma_{b_s}$ is the subgroup of permutations from Σ_r preserving the groups of repeating symbols among $j_1, \ldots, j_r$.

The vector space

$$\mathrm{Sym}(E) := \bigoplus_{r \ge 0} S_r E$$

has a natural multiplication

$$m : \mathrm{Sym}(E) \otimes \mathrm{Sym}(E) \to \mathrm{Sym}(E)$$

given by the formula

$$m(u_1 \dots u_r \otimes v_1 \dots v_s) = u_1 \dots u_r v_1 \dots v_s.$$

This gives $\mathrm{Sym}(E)$ the structure of associative, commutative algebra. We call this algebra *the symmetric algebra on* E. It can be identified with the polynomial ring over $\mathbf{K}$ in n variables $e_1, \dots, e_n$. In order to keep the notion of commutativity the same as for the exterior algebras, we assume that $\mathrm{Sym}(E)$ is generated by elements of degree 2.

The components of the multiplication map will be denoted by $m : S_r E \otimes S_s E \to S_{r+s} E$.

We also have an obvious unit map $\eta : \mathbf{K} \to \mathrm{Sym}(E)$ sending $\mathbf{K}$ to the degree zero component of $\mathrm{Sym}(E)$.

The diagonal map $\Delta : E \to E \oplus E$ induces an algebra map

$$\Delta : \mathrm{Sym}(E) \to \mathrm{Sym}(E \oplus E) \cong \mathrm{Sym}(E) \otimes \mathrm{Sym}(E),$$

which we will call *the diagonal (or comultiplication) map*.

The components of Δ will be denoted by $\Delta : S_{r+s} E \to S_r E \otimes S_s E$. In terms of elements we have

$$\Delta(u_1 \dots u_{r+s}) = \sum_{\sigma \in \Sigma^{r,s}_{r+s}} u_{\sigma(1)} \dots u_{\sigma(r)} \otimes u_{\sigma(r+1)} \dots u_{\sigma(r+s)}$$

where $\Sigma^{r,s}_{r+s} = \{\sigma \in \Sigma_{r+s} \mid \sigma(1) < \dots < \sigma(r); \sigma(r+1) < \dots < \sigma(r+s)\}$.

Finally we have the counit map

$$\epsilon : \mathrm{Sym}(E) \to \mathbf{K}$$

defined to be zero on all spaces $S_r E$ for $r > 0$, and satisfying $\epsilon\eta(1) = 1$.

We have the following analogue of (1.1.2) (a).

(1.1.4) Proposition. *The maps* $m, \Delta, \epsilon, \eta$ *define on* $\mathrm{Sym}(E)$ *the structure of a commutative, cocommutative bialgebra.*

Let us investigate the duality. The algebra $\mathrm{Sym}(E) = \bigoplus_{r \geq 0} S_r E$ is not finite dimensional, so instead of the dual we have to work with the graded dual

$$\mathrm{Sym}(E)^*_{gr} := \bigoplus_{r \geq 0} (S_r E)^*.$$

The module map

$$E^* = (S_1E)^* \to \mathrm{Sym}(E)^*_{gr}$$

induces by universality an algebra map

$$\beta : \mathrm{Sym}(E^*) \to \mathrm{Sym}(E)^*_{gr}.$$

This map β is an isomorphism only when $\mathbf{K}$ contains a field of rational numbers. In fact it is given by the formula

$$\beta(l_1 \dots l_r)(u_1 \dots u_r) = \sum_{\sigma \in \Sigma_r} l_{\sigma(1)}(u_1) \dots l_{\sigma(r)}(u_r).$$

In particular, when $l_1 = \dots = l_r,\ u_1 = \dots = u_r$ we see that $\beta(l_1^r) = r!(u_1^r)^*$.

In order to describe the graded dual of the symmetric algebra we introduce the divided powers.

We define *the r-th divided power* $D_r(E)$ as the dual of the symmetric power.

$$D_r(E) := (S_r(E^*))^*.$$

Its basis is the dual basis to the natural basis of the symmetric power. If $\{e_1, \dots, e_n\}$ is a basis of E, we define $e_1^{(i_1)} \dots e_n^{(i_n)}$ to be the element of the dual basis to the basis $\{(e_1^*)^{j_1} \dots (e_n^*)^{j_n}\}$, dual to $(e_1^*)^{i_1} \dots (e_n^*)^{i_n}$.

For every $u \in E$ we can define its *r-th divided power* $u^{(r)} \in D_rE$. It is given by the formula

$$\left(\sum_{i=1}^n u_i e_i\right)^{(r)} = \sum_{p_1+\dots+p_n=r} u_1^{p_1} \dots u_n^{p_n} e_1^{(p_1)} \dots e_n^{(p_n)}.$$

It is easy to check that this definition does not depend on the choice of basis $\{e_1, \dots, e_n\}$.

(1.1.5) Proposition. *The divided powers have the following properties:*

(a) $u^{(0)} = 1,\ u^{(1)} = u,\ u^{(r)} \in D_rE,$
(b) $u^{(p)}u^{(q)} = \binom{p+q}{q} u^{(p+q)},$
(c) $(u+v)^{(p)} = \sum_{k=0}^{p} u^{(k)} v^{(p-k)},$
(d) $(uv)^{(p)} = u^{(p)} v^{(p)},$
(e) $(u^{(p)})^{(q)} = [p,q] u^{(pq)}$ *for* $u \in E$*;* $[p,q] = [(pq)!]/(q!p^q!)$.

(1.1.6) Remark. *In the notation used above,* $e_1^{(i_1)} \dots e_n^{(i_n)}$ *has a double meaning. It is the element of the dual basis to the basis in the symmetric power,* *and it is the product of divided powers. It is not difficult to see that the two elements coincide.*

The r-th divided power is an endofunctor on the category of free $\mathbf{K}$-modules and linear maps. More precisely, for two free $\mathbf{K}$-modules E, F and a linear map $\phi : E \to F$ we have a well-defined linear map

$$D_r\phi : D_r E \to D_r F$$

which is best described as the transpose of the map $S_r(\phi^*) : S_r(F^*) \to S_r(E^*)$. This also gives the description of the matrix coefficients for $D_r\phi$ as polynomials in the entries of ϕ, which we leave to the reader.

The divided power algebra $D(E) := \bigoplus D_r(E)$ on E is a commutative, cocommutative algebra because it is a graded dual of the symmetric algebra on E^*. Again we denote the components of the multiplication map by

$$m : D_r E \otimes D_s E \to D_{r+s} E,$$

and the components of the comultiplication by

$$\Delta : D_{r+s} E \to D_r E \otimes D_s E.$$

Let us record the duality statements.

(1.1.7) Proposition.

(a) The multiplication map

$$m : D_r E \otimes D_s E \to D_{r+s} E$$

is the dual of the diagonal map

$$\Delta : S_{r+s} E^* \to S_r E^* \otimes S_s E^*.$$

(b) The diagonal map

$$\Delta : D_{r+s} E \to D_r E \otimes D_s E$$

is the dual of the multiplication map

$$m : S_r E^* \otimes S_s E^* \to S_{r+s} E^*.$$

(c) The diagonal map $\Delta : D_{r+s} E \to D_r E \otimes D_s E$ is given by the formula

$$\Delta(e_1^{(i_1)} \dots e_n^{(i_n)}) = \sum_{j_1+\dots+j_n=r,\quad 0\le j_s\le i_s \text{ for } s=1,\dots,n} e_1^{(j_1)} \dots e_n^{(j_n)} \otimes e_1^{(i_1-j_1)} \dots e_n^{(i_n-j_n)}.$$

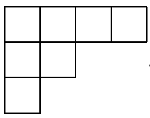
1.1.2. Partitions, Skew Partitions. Combinatorics of $\mathbf{Z}_2$-Graded Tableaux.

Let n be a natural number. *A partition* λ of n is a sequence $\lambda = (\lambda_1, \dots, \lambda_s)$ of natural numbers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s \geq 0$ and $\lambda_1 + \lambda_2 + \dots + \lambda_s = n$. We identify the partitions $(\lambda_1, \dots, \lambda_s)$ and $(\lambda_1, \dots, \lambda_s, 0)$. To each partition λ we associate its Young frame (or Ferrers diagram) $D(\lambda)$. It can be defined as

$$D(\lambda) = \{(i, j) \in \mathbf{Z} \times \mathbf{Z} \,|(1 \leq i \leq s, 1 \leq j \leq \lambda_i \}.$$

To represent the Young frames graphically we think of them as contained in the fourth quadrant. A Young frame is a set of boxes with λ_i boxes in the i-th row from the top. Formally it could be achieved by considering the point $(j, -i)$ instead of (i, j).

(1.1.8) Example. $\lambda = (4, 2, 1)$:

$D(\lambda) =$  .

Formally the boxes of $D((4, 2, 1))$ *correspond to the set of points*

$$\{(1, -1), (2, -1), (3, -1), (4, -1), (1, -2), (2, -2), (1, -3)\}$$

in the grid $\mathbf{Z} \times \mathbf{Z}$.

Let λ be a partition. We say that λ has *a Durfee square of size* r (or rank $\lambda = r$) if $\lambda_r \geq r$, $\lambda_{r+1} \leq r$, i.e., if the biggest square fitting inside of λ is an $r \times r$ square.

Let λ be a partition, and let X be a box in λ. The set of boxes to the right of X (including X) is called an *arm of* X. The set of boxes below X (including X) is called the *leg of* X. The arm length (leg length) of X are defined as the numbers of boxes in the arm (leg) of X.

The arm and leg of X form *a hook* of X. The number of boxes in the hook of X is called the hook length of X.

Let λ be a partition of rank r. Let a_i (b_i) be the arm length (leg length) of the i-th box on the diagonal of λ. The partition λ is uniquely determined by its rank r and the numbers a_i, b_i $(1 \leq i \leq r)$. These numbers satisfy the conditions $a_1 > \dots > a_r > 0,\ b_1 > \dots,\ b_r > 0$.

We will sometimes denote by $\lambda = (a_1, \ldots, a_r | b_1, \ldots, b_r)$ the partition with diagonal arm lengths a_i and diagonal leg lengths b_i. We refer to this as a *Frobenius* (or *hook*) *notation* for λ.

(1.1.9) Example. *The partition* $\lambda = (4, 3, 2)$ *in the hook notation is* $(4, 2|3, 2)$. *The boxes in the arm (leg) of the i-th diagonal box are filled with symbol i* ($\bar{i}$)*:*

X	1	1	1
$\bar{1}$	X	2	
$\bar{1}$	$\bar{2}$		

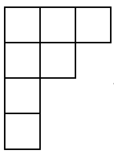
.

Let λ be a partition. The conjugate (or dual) partition λ' is defined by setting

$$\lambda'_i = \text{card}\{t \mid \lambda_t \geq i\}.$$

The Young frame of λ' is obtained from the Young frame of λ by reflecting in the line $y = -x$.

(1.1.10) Example. $\lambda = (4, 2, 1)$, $\lambda' = (3, 2, 1, 1)$:

$D(\lambda') =$  .

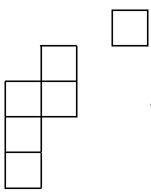
Let λ and μ be two partitions. We say that μ is contained in λ (denoted $\mu \subset \lambda$) if for each i we have $\mu_i \leq \lambda_i$. Let λ and μ be two partitions with $\mu \subset \lambda$. We refer to such a pair as a *skew partition* λ/μ.

We associate to a skew partition λ/μ the skew Young frame

$$D(\lambda/\mu) := D(\lambda) \setminus D(\mu).$$

Graphically we can represent it as a Young frame of λ with the boxes corresponding to μ missing.

(1.1.11) Example. $\lambda = (4, 2, 2, 1, 1)$, $\mu = (3, 1)$:

$D(\lambda/\mu) =$  .

Let $A = (A_0, A_1)$ be a $\mathbf{Z}_2$-graded set, i.e. the pair of sets indexed by $\{0, 1\}$. Assume that the set A is ordered by a total order $\triangleleft$. A *tableau of shape* λ/μ *with values in* A is a function $T : D(\lambda/\mu) \to A$.

(1.1.12) Definition.

(a) A tableau T of shape λ/μ with values in A is row standard if for each (u, v) we have $T(u, v) \triangleleft T(u, v+1)$ with equality possible if $T(u, v) \in A_1$.

(b) We say that a tableau T of shape λ/μ with values in A is column standard if $T(u, v) \triangleleft T(u+1, v)$ with equality possible when $T(u, v) \in A_0$.

(c) A tableau T of shape λ/μ with values in A is standard if it is both column standard and row standard.

(1.1.13) Notation. *We denote by* $\mathrm{RST}(\lambda/\mu, A)$ *(*$\mathrm{CST}(\lambda/\mu, A)$, $\mathrm{ST}(\lambda/\mu, A)$*) the set of row standard (column standard, standard) tableaux of shape λ/μ with values in A. We denote by $[1, m] \cup [1, n]'$ the $\mathbf{Z}_2$-graded set A with $A_0 = [1, m]$, $A_1 = [1', n']$ and with the order $\triangleleft$ defined to be the natural order on A_0 and A_1 with A_0 preceeding A_1. Similarly we define the $\mathbf{Z}_2$-graded set $[1, n]' \cup [1, m]$ (here A_1 preceeds A_0).*

(1.1.14) Examples. *Let $\lambda = (4, 2, 2, 1, 1)$, $\mu = (2, 1)$. Let $A = [1, 2] \cup [1, 3]'$.*

(a) The tableau

$$T_1 = \begin{array}{cccc} & & \boxed{1} & \boxed{2} \\ & \boxed{1'} & & \\ \boxed{1'} & \boxed{1'} & & \\ \boxed{2'} & & & \\ \boxed{2'} & & & \end{array}$$

is row standard but not column standard.

(b) The tableau

$$T_2 = \begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{1'} & & \\ \boxed{1'} & \boxed{2'} & & \\ \boxed{2'} & & & \\ \boxed{3'} & & & \end{array}$$

is column standard but not row standard.