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Part One

Fourier Series and Periodic Distributions

1

Preliminaries

In this chapter we present some basic definitions and some of the problems and concepts that will be discussed and used throughout the book. The material presented here is by no means a complete account of such topics as classification into types, canonical forms, the method of characteristics and so on. There are several excellent accounts of these in the literature (see [57], [60], [64], [86] and [151] for example).

1.1 Basic Definitions and Examples

Let us begin by introducing some notation and terminology. An **open ball** of radius $r > 0$ centered at $x_0 \in \mathbb{R}^n$ is a set of the form

$$B(x_0;r) = \{x \in \mathbb{R}^n\colon\ |x - x_0| < r\},$$

where x_0 is a fixed point in $\mathbb{R}^n$, $|\cdot|$ is the usual euclidean norm in $\mathbb{R}^n$ and r is a positive real number. Similarly, a **closed ball** in $\mathbb{R}^n$ is a set of the form $\overline{B(x_0;r)} = \{x \in \mathbb{R}^n : |x - x_0| \leqslant r\}$. A subset $\Omega \subseteq \mathbb{R}^n$ is said to be **open** if, for any $x \in \Omega$, there exists an open ball $B(x;r)$ contained in Ω. A subset $K \subseteq \mathbb{R}^n$ is **closed** if its complement $\mathbb{R}^n \setminus K = \{x \in \mathbb{R}^n : x \notin K\}$ is open. The **closure** of $S \subseteq \mathbb{R}^n$, denoted by $\overline{S}$, is the smallest closed set containing S, i.e., $\overline{S} = \bigcap\{K \subseteq \mathbb{R}^n : K \text{ is closed and } S \subseteq K\}$. The **interior** of $S \subseteq \mathbb{R}^n$, denoted by $\operatorname{Int}(S)$, is the largest open set contained in S, that is, $\operatorname{Int}(S) = \bigcup\{\Omega \subseteq \mathbb{R}^n : \Omega \text{ is open and } \Omega \subseteq S\}$. The **boundary** of $S \subseteq \mathbb{R}^n$ is the set $\partial S = \overline{S} \cap \overline{(\mathbb{R}^n \setminus S)}$. It is easy to see that the closed ball $\overline{B(x_0;r)}$ is in fact the closure of the open ball $B(x_0;r)$, that the interior of the closed ball $\overline{B(x_0;r)}$ is the open ball $B(x_0;r)$ and that the boundary of both the open and the closed balls is the **sphere** $\{x \in \mathbb{R}^n : |x - x_0| = r\}$. An open subset $\Omega \subseteq \mathbb{R}^n$ is **connected** if there are no disjoint nonempty open sets Ω_1 and Ω_2 in $\mathbb{R}^n$ such that $\Omega = \Omega_1 \cup \Omega_2$. An open connected subset of

$\mathbb{R}^n$ is called a **domain**. As usual, if $\Omega \subseteq \mathbb{R}^n$ is an open subset, we denote by $C^k(\Omega)$ the set of all functions $\Omega \to \mathbb{C}$ that are k times continuously differentiable. The **support** of a function $f : \Omega \longrightarrow \mathbb{C}$, denoted by $supp(f)$, is the smallest closed set outside which f vanishes identically. We use the notation $C_0^k(\Omega)$ for the set of all functions $\Omega \to \mathbb{C}$ that are k times continuously differentiable and have compact support in Ω. The set of all complex valued infinitely differentiable functions on Ω is denoted by $C^\infty(\Omega)$ and the set of all complex valued infinitely differentiable functions with compact support in Ω is denoted by $C_0^\infty(\Omega)$. If $[a,b] \subseteq \mathbb{R}$ is a closed interval, $C^k([a,b])$ is the set of all functions $f : [a,b] \to \mathbb{C}$ that are k times differentiable in the closed interval with the kth derivative $f^{(k)} \in C([a,b])$; the differentiability at the endpoints is defined by the one-sided limits

$$f'(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h},$$
$$f'(b) = \lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h}.$$

We define in a similar way $C^k([a,\infty))$ and $C^k((-\infty,b])$, where $a,b \in \mathbb{R}$. For infinitely differentiable functions we will use the notations $C^\infty([a,b])$, $C^\infty([a,\infty))$ and $C^\infty((-\infty,b])$.

A **differential equation (DE)** is an equation involving one or more independent variables, an unknown function, and its derivatives with respect to these variables. If there is only one independent variable x, we say that the equation is an **ordinary differential equation (ODE).** If there are two or more independent variables, $x_1, x_2, \ldots, x_n$, we say that the equation is a **partial differential equation (PDE).** Thus, an ODE is an expression of the form

$$F\left(x, u, u', \ldots, u^{(m)}\right) = 0 \tag{1.1}$$

where $u', \ldots, u^{(m)}$ denote the derivatives of $u(x)$ with respect to x up to order m in some open subset of $\mathbb{R}$, while a PDE has the form

$$G\left(x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^m u}{\partial x_n^m}\right) = 0, \tag{1.2}$$

where $x = (x_1, x_2, \ldots, x_n)$ belongs to some open set $\Omega \subseteq \mathbb{R}^n$, F and G are given functions, u is to be determined and

$$\frac{\partial^j u}{\partial x_1^{j_1} \partial x_2^{j_2} \ldots \partial x_n^{j_n}}, \quad j = j_1 + j_2 + \cdots + j_n,$$

denotes the jth order partial derivative of u. We will often use the following alternative notations.

$$\partial_{x_j}^k u = \frac{\partial^k u}{\partial x_j^k} = u_{\underbrace{x_j \dots x_j}_{k \text{ times}}}.$$

The above definitions are too general. It is easy to devise very strange and useless equations like

$$\exp\left(u'(x)\right) = 0$$

or

$$\frac{1}{(u'(x))^2 + u(x)} = 0.$$

Thus, it is important to determine which equations are meaningful and restrict one's attention to those subclasses. In the remainder of this section we will exhibit several examples of interesting equations that will be considered in the course of the book.

The **order** of a partial differential equation is the order of the highest order derivative occurring in the equation. If F and G are not constant, when considered as functions of the derivatives of order m, then both (1.1) and (1.2) have order m. A partial differential equation is **linear** if it is a polynomial of the first degree in u and its derivatives. Otherwise, the PDE is **nonlinear.** The most general second order linear PDE has the form

$$\sum_{j,k=1}^{n} a_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x_j} + c(x) u(x) + d(x) = 0, \qquad (1.3)$$

where at least one of the coefficients $a_{jk}(x)$ is not identically zero. If $d = 0$, we say that (1.3) is **homogeneous;** otherwise (1.3) is **nonhomogeneous.** The **principal part** of a PDE is the part of the equation that contains the derivatives of highest order. In the case of (1.3), the principal part is the double sum on the left hand side. Nonlinear equations with linear principal parts are called **semilinear.** The general second order semilinear PDE is

$$\sum_{j,k=1}^{n} a_{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} = f\left(x, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right).$$

The most important examples of linear equations are the following.

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EXAMPLE 1.1. The three classical equations of mathematical physics, as follows.

• The **heat equation**

$$\partial_t u(t,x) = \alpha^2 \Delta u(t,x), \ t > 0, \ x \in \Omega \subseteq \mathbb{R}^n, \tag{1.4}$$

where α^2 is a constant, known as the **diffusion coefficient**, and

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \tag{1.5}$$

is the **Laplacian** (or **Laplace operator**) in $\mathbb{R}^n$. This equation is associated with diffusion phenomena, like the flow of heat in a conducting medium (see [60], [114], [151] and [162]).

• The **wave equation**

$$\partial_t^2 u(t,x) = c^2 \Delta u(t,x), \ t > 0, \ x \in \Omega \subseteq \mathbb{R}^n. \tag{1.6}$$

This equation describes wave phenomena, like the motion of a membrane or waves traveling in a string. Here c is a positive constant, known as the **speed of propagation** of the wave (see [60], [114], [125] and [151]).

• **Laplace's equation**

$$\Delta u(x) = 0, \ x \in \Omega \subseteq \mathbb{R}^n. \tag{1.7}$$

This equation describes stationary phenomena, such as the electrostatic potential generated by fixed distributions of electric charges (see [60], [83], [114], [151] and [155], for example). Note that the stationary (i.e. time independent) solutions of the heat and wave equations satisfy Laplace's equation. Functions satisfying (1.7) are said to be **harmonic** in Ω.

EXAMPLE 1.2. The nonhomogeneous versions of the equations in Example 1.1, that is,

$$\partial_t u(t,x) = \alpha^2 \Delta u(t,x) + f(t,x), \ t > 0, \ x \in \Omega \subseteq \mathbb{R}^n, \tag{1.8}$$

$$\partial_t^2 u(t,x) = c^2 \Delta u(t,x) + g(t,x), \ t > 0, \ x \in \Omega \subseteq \mathbb{R}^n, \tag{1.9}$$

$$\Delta u(x) = h(x), \ x \in \Omega \subseteq \mathbb{R}^n, \tag{1.10}$$

where f, g, and h are given functions. Equation (1.10) is known as **Poisson's equation**.

EXAMPLE 1.3. **Schrödinger's equation**

$$i\partial_t u(t,x) = -\frac{\hbar}{2m} \Delta u(t,x) + V(x)\, u(t,x), \ t > 0, \ x \in \Omega \subseteq \mathbb{R}^n, \tag{1.11}$$

where $V(x)$ is a 'well-behaved' real valued function, $h = 2\pi\hbar = 6.625 \times 10^{-34}$ is Planck's constant and m is a positive constant. When $\Omega = \mathbb{R}^3$, this equation describes the motion of a quantum mechanical particle of mass m interacting with the potential $V(x)$. Equation (1.11) is one of the most popular equations of this century. It has been exhaustively studied by many authors during the last sixty years. See [132]–[134] for the state of the art at the beginning of the 1980s; see also [19], [38], [49], [51], [70], [84], [113] and [144].

On the nonlinear level, there are many interesting equations. Among those discussed in this book we find:

EXAMPLE 1.4. The **nonlinear Schrödinger equation (NLS)**

$$i\partial_t u(t,x) = -\Delta u(t,x) + |u(t,x)|^{p-1} u(t,x), \; t > 0, \, x \in \Omega \subseteq \mathbb{R}^n, \qquad (1.12)$$

where $p \geqslant 2$ is an integer. This equation occurs, for instance, in plasma physics (see [116] and [161]). There is a very rich theory associated with it. For a great deal of information that cannot be supplied in this text, the interested reader should consult [41], [42] and [58].

EXAMPLE 1.5. **Burgers' equation**

$$\partial_t u(t,x) + u(t,x)\,\partial_x u(t,x) = \mu \partial_x^2 u(t,x), t > 0, \; x \in \mathbb{R}, \, \mu \geqslant 0. \qquad (1.13)$$

Note that if $\mu > 0$, we can regard (1.13) as a nonlinear perturbation of the heat equation. For further information see [53] and [161].

EXAMPLE 1.6. Equations describing the motion of waves in fluids such as the following.
The **Korteweg–de Vries equation (KdV)**

$$u_t(t,x) + u_{xxx}(t,x) + u(t,x)\,u_x(t,x) = 0, \; t > 0, \; x \in \mathbb{R}. \qquad (1.14)$$

This equation describes the motion of internal gravity waves in shallow channels. It has been the object of intense research recently due to its very special properties (which, it turned out, are shared by a very large class of equations). Some of these properties will be discussed and used in Chapter 6 to establish certain characteristics of its solutions. For further information on this subject see [91], [107], [109], [116], [117] and [161].
The **Benjamin–Bona–Mahony equation (BBM)**

$$\partial_t u(t,x) + \partial_x^2 \partial_t u(t,x) + u(t,x)\,\partial_x u(t,x) = 0. \qquad (1.15)$$

This equation is an alternative model for the class of phenomena described by KdV. It was first obtained and studied in [21]. Since then a lot more has been written about it (see [6], [12] and [13], for instance).

The **Benjamin–Ono equation (BO)**

$$u_t(t,x) + \sigma\partial_x^2 u(t,x) + u(t,x)\,u_x(t,x) = 0, \;\; t > 0, \;\; x \in \mathbb{R}, \tag{1.16}$$

where σ denotes the Hilbert transform,

$$(\sigma f)(x) = p.v.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(y)}{y-x}\,dy. \tag{1.17}$$

This equation describes the propagation of waves in a stratified fluid of infinite depth. It is very similar to KdV and shares many of its properties. However, there are some rather remarkable differences between the two (see Chapters 6 and 8). More information on BO can be found in [20], [52], [80], [120], and [124].

The behavior of a PDE is very different from that of an ODE. On the one hand, if it is possible to get a general solution of a PDE (a rare event!), it involves arbitrary functions of the independent variables rather than constants, as is the case with an ODE, so there is a greater freedom with respect to the *form* of the solution. On the other hand, very simple linear PDEs may fail to have a solution: Hans Lewy, in 1957, showed that there exists a C^∞ real function $f = f(t)$ such that the nonhomogeneous linear equation

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} - 2i(x+iy)\frac{\partial u}{\partial t} = f(t)$$

has no solution (not even in the sense of distribution theory; see [60]).

Up to now we have not defined what we mean by a *solution* of a PDE. The intuitive notion that a solution is a function that satisfies the equation identically is not precise: we will see later that there are many possible interpretations of this intuitive notion (even generalizing the concept of function). At the moment we will consider the so called classical solutions: a **classical solution** of a PDE of order m in a domain $\Omega \subseteq \mathbb{R}^n$ is a function $u \in C^m(\Omega)$ that satisfies the equation at all points of Ω. If we are only interested in classical solutions in the case $m \geqslant 2$, then $\partial_{x_i}\partial_{x_j}u = \partial_{x_j}\partial_{x_i}u$. Thus, in the case of two independent variables, it is usual to write a semilinear PDE of order 2 in the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \tag{1.18}$$

To illustrate the concept of solution and the remark about a general solution of a PDE, let us look at the homogeneous wave equation with one spatial dimension,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, (t,x) \in \mathbb{R}^2. \tag{1.19}$$

The general classical solution of (1.19) has the form

$$u(t,x) = f(x+ct) + g(x-ct), \tag{1.20}$$

where $f, g \in C^2(\mathbb{R})$ are arbitrary. It is clear that if $f, g \in C^2(\mathbb{R})$ then u, defined by (1.20), is a classical solution of (1.19). To prove that all classical solutions have this form, introduce the change of variables

$$\begin{aligned} \xi &= x + ct, \\ \eta &= x - ct, \\ v(\xi,\eta) &= u(t,x). \end{aligned} \tag{1.21}$$

It is clear that v is also a C^2 function. An application of the chain rule leads to

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = 0.$$

Integrating first with respect to ξ and then with respect to η, we get

$$v(\xi,\eta) = f(\xi) + g(\eta),$$

where $f, g \in C^2(\mathbb{R})$ are arbitrary. Going back to the original variables, we arrive at (1.20).

The reader shouldn't be alarmed by the 'magical' method used above. In the next section we will justify the change of variables (1.21) and in Section 5 we will obtain (1.20) by another method.

Exercise 1.7. Find the order of each of the following equations.

(a) $\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial^3 u}{\partial y^3} = 0.$
(b) $u\partial_x^2\partial_y u + \partial_x u = u^2 + 1.$
(c) $u_x u_t = \sin u.$
(d) $x^3\partial_x u - u^3\partial_t u + \partial_x^2 u = x^5 + t^4.$
(e) $\frac{\partial}{\partial x}\left(u^2\right) - \frac{\partial u}{\partial y} = xyu.$

Exercise 1.8. Indicate which of the following equations are linear and, for the linear ones, which of them are homogeneous.

(a) $\partial_x\left(u^2\right) + \partial_y u = 0.$

(b) $x^3 \frac{\partial^2 u}{\partial x \partial y} - y^2 \frac{\partial u}{\partial y} + u = x + y$.
(c) $(u_x)^2 - x^2 + u_t = 0$.
(d) $u_{xx} - u_{tt} = \sin u$.
(e) $x^2 \partial_y^2 u + y^2 \partial_x^2 u = \partial_x u + \partial_y u + xyu$.

Exercise 1.9. Identify the principal part of each equation in Exercises 1.7 and 1.8.

Exercise 1.10. Indicate which of the equations in Exercises 1.7 and 1.8 are semilinear.

Exercise 1.11. For each of the following equations, prove directly that if u and v are solutions then any linear combination of u and v is also a solution.

(a) $\frac{\partial u}{\partial x} + xu = 0$.
(b) $y \frac{\partial^2 u}{\partial x^2} = 0$.
(c) $xu_y = xyu$.
(d) $yu_{xx} + xu_y = xyu$.
(e) $yu_{xx} + u_x + xu = 0$.

Exercise 1.12. (a) Let $\mu > 0$ be fixed. Show that, in this case, Burgers' equation (see (1.13)) transforms into the heat equation

$$v_t = \mu v_{xx} \tag{1.22}$$

under the following change of dependent variable:

$$u = -2\mu \frac{v_x}{v}. \tag{1.23}$$

Formula (1.23) is known as the Cole–Hopf transformation. It can be used to obtain explicit solutions of (1.13), as shown below, and to completely analyze its behavior, even in the limit as $\mu \searrow 0$. For further details see Chapter 4 of [161].

(b) Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and bounded. Prove that

$$v(t,x) = \left(\frac{1}{4\pi\mu t}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{|x-y|^2}{4\mu t}\right) \psi(y)\, dy \tag{1.24}$$

solves (1.22).

(c) It is possible to show that (1.24) is the unique solution of the Cauchy problem

$$\begin{aligned} v &\in C([0,\infty)\times\mathbb{R})\cap C^2((0,\infty)\times\mathbb{R}),\\ v_t &= \mu v_{xx},\\ v(0,x) &= \psi(x). \end{aligned}$$

Use this information to show that one should expect that the Cauchy problem associated to Burgers' equation,

$$\begin{aligned} \partial_t u(t,x) + u(t,x)\,\partial_x u(t,x) &= \mu\partial_x^2 u(t,x),\\ u(0,x) &= \phi(x), \end{aligned}$$

must have a solution of the form

$$u(x,t) = \left(\frac{1}{4\pi\mu t}\right)^{\frac{1}{2}} \frac{1}{G(t,x)} \int_{\mathbb{R}} \exp\left(-\frac{|x-y|^2}{4\mu t}\right) G_0(y)\,\phi(y)\,dy \qquad (1.25)$$

where

$$G(t,x) = \left(\frac{1}{4\pi\mu t}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp\left(-\frac{|x-y|^2}{4\mu t}\right) G_0(y)\,dy$$

and

$$G_0(x) = -\frac{1}{2\mu}\int_{-\infty}^{x} \phi(y)\,dy.$$

Show that (1.25) is indeed a solution if we assume that $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, bounded and integrable in the sense that

$$\int_{\mathbb{R}} |f(y)|\,dy = \lim_{\substack{a\to-\infty\\ b\to\infty}} \int_a^b |f(y)|\,dy < \infty.$$

EXERCISE 1.13. Let $\psi\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a C^3 function and $c \neq 0$ a real constant.

(a) Prove that $u(t,x) = \psi(x-ct)$ is a solution of the Korteweg–de Vries equation (1.14) if and only if

$$\psi'' + \frac{1}{2}\psi^2 - c\psi = A,$$

where A is an integration constant. Multiply this equation by $2\psi'$ and integrate to show that

$$(\psi')^2 + \frac{1}{3}\psi^3 - c\psi^2 = 2A\psi + B$$

where B is another integration constant.