
Contents

	<i>Introduction</i>	<i>page xi</i>
I	The Gauss–Manin connection	1
1	Milnor fibration, Picard–Lefschetz monodromy transformation, topological Gauss–Manin connection	1
	1.1 Milnor fibration	1
	1.2 Cohomological Milnor fibration	1
	1.3 Topological Gauss–Manin connection	2
	1.4 Picard–Lefschetz monodromy transformation	2
2	Connections, locally constant sheaves and systems of linear differential equations	3
	2.1 Connection as a covariant differentiation	3
	2.2 Equivalent definition: a covariant derivative along a vector field	4
	2.3 Local calculation of connections. Relation to differential equations	5
	2.4 The integrable connections. The De Rham complex	6
	2.5 Local systems and integrable connections	7
	2.6 Dual local systems and connections	8
3	De Rham cohomology	10
	3.1 The Poincaré lemma	10
	3.2 Relative De Rham cohomology	11
	3.3 De Rham cohomology for smooth Stein morphisms	11
	3.4 Coherence theorem	12
	3.5 On the absence of torsion in the De Rham cohomology sheaves	12
	3.6 Relation between $\mathcal{H}^p(f_*\Omega_f)$ and $f_*\mathcal{H}^p(\Omega_f)$	13
4	Gauss–Manin connection on relative De Rham cohomology	14

vi	<i>Contents</i>	
	4.1 Identification of sheaves of sections of cohomological fibration and of relative De Rham cohomology	15
	4.2 Calculation of the connection on a relative De Rham cohomology sheaf	16
	4.3 The division lemma. The connections on the sheaves $\mathcal{H}_{\text{DR}}^p(X/S)$ for $p \leq n-1$	17
	4.4 The sheaf $'\mathcal{H} = f_*\Omega_{X/S}^n/d(f_*\Omega_{X/S}^{n-1})$	19
	4.5 Meromorphic connections	20
	4.6 The Gauss–Manin connection as a connecting homomorphism	21
5	Brieskorn lattices	23
	5.1 Brieskorn lattice $''\mathcal{H}$	24
	5.2 Calculation of the Gauss–Manin connection ∇ on $'\mathcal{H}$	25
	5.3 Increasing filtration on $\mathcal{H}^{(0)}$	25
	5.4 A practical method of calculation of the Gauss–Manin connection	27
	5.5 Calculation of the Gauss–Manin connection of quasi-homogeneous isolated singularities	28
6	Absence of torsion in sheaves $\mathcal{H}^{(-i)}$ of isolated singularities	30
	6.1 The presence of a connection implies the absence of torsion	30
	6.2 A theorem of Malgrange	31
	6.3 Connection on a pair (E, F)	32
	6.4 Sheaves $\mathcal{H}^{(-p)}$ are locally free	32
7	Singular points of systems of linear differential equations	33
	7.1 Differential equations of Fuchsian type	33
	7.2 Systems of linear differential equations and connections	34
	7.3 Decomposition of a fundamental matrix $Y(t)$	35
	7.4 Regular singular points	36
	7.5 Simple singular points	36
	7.6 Simple singular points are regular	37
	7.7 Connections with regular singularities	39
	7.8 Residue and limit monodromy	41
8	Regularity of the Gauss–Manin connection	42
	8.1 The period matrix and the Picard–Fuchs equation	42
	8.2 The regularity theorem follows from Malgrange’s theorem	44
	8.3 The regularity theorem and connections with logarithmic poles	44
9	The monodromy theorem	46
	9.1 Two parts of the monodromy theorem	46

<i>Contents</i>		vii
9.2	Eigenvalues of monodromy	47
9.3	The size of Jordan blocks	49
9.4	Consequences of the monodromy theorem. Decomposition of integrals into series	49
10	Gauss–Manin connection of a non-isolated hypersurface singularity	51
10.1	De Rham cohomology sheaves	51
10.2	Coherence	52
10.3	Relation between $\mathcal{H}^p(f_*\Omega_f)$ and $f_*\mathcal{H}^p(\Omega_f)$	53
10.4	A general method of extension of a singular connection over the whole disk	53
10.5	The sheaves $\mathcal{H}_{(-1)}^p$ and the Gauss–Manin connection $\partial_t: \mathcal{H}_{(-2)}^p \Rightarrow \mathcal{H}_{(-1)}^p$	54
10.6	The sheaves $\mathcal{H}_{(0)}^p$ and the Gauss–Manin connection $\partial_t: \mathcal{H}_{(-1)}^p \Rightarrow \mathcal{H}_{(0)}^p$	56
10.7	A generalization of diagram (5.3.4)	57
II	Limit mixed Hodge structure on the vanishing cohomology of an isolated hypersurface singularity	60
1	Mixed Hodge structures. Definitions. Deligne’s theorem	60
1.1	Pure Hodge structure	60
1.2	Polarised HSs	61
1.3	Mixed Hodge structure	61
1.4	Deligne’s theorem	62
2	The limit MHS according to Schmid	62
2.1	Variation of HS: geometric case	62
2.2	Variation of HS: definition	63
2.3	Classifying spaces and period mappings	63
2.4	The canonical Milnor fibre	64
2.5	The Schmid limit Hodge filtration $F_{\mathfrak{S}}$	67
2.6	An interpretation of $F_{\mathfrak{S}}$ in terms of the canonical extension of \mathcal{H}	69
2.7	The weight filtration of a nilpotent operator	70
2.8	Schmid’s theorem	73
3	The limit MHS according to Steenbrink	73
3.1	The limit MHS for projective families: the case of unipotent monodromy	74
3.2	The limit MHS for projective families: the general case	75
3.3	Brieskorn construction	77
3.4	Limit MHS on a vanishing cohomology	78
3.5	The weight filtration on $H^n(X_\infty)$. Symmetry of Hodge numbers	79

viii	<i>Contents</i>	
4	Hodge theory of a smooth hypersurface according to Griffiths–Deligne	82
4.1	The Gysin exact sequence	82
4.2	Hodge theory for a complement $U = X \setminus Y$. Hodge filtration and pole order filtration	83
4.3	De Rham complex of the sheaf $B_{[Y]X}$ and the cohomology of a hypersurface Y	85
4.4	The case of a smooth hypersurface Y in a projective space $X = \mathbb{P}^{n+1}$	86
4.5	Generalization to the case of a hypersurface with singularities	87
5	The Gauss–Manin system of an isolated singularity	88
5.1	Hodge theory of a smooth hypersurface in the relative case	89
5.2	The Gauss–Manin differential system	90
5.3	Interpretation of the complex $DR_{Z/S}^*(B_{[\Gamma]Z})$ in terms of the morphism $f: X \rightarrow S$	91
5.4	Connection between the differential system \mathcal{H}_X and the Brieskorn lattice $\mathcal{H}^{(0)}$	94
6	Decomposition of a meromorphic connection into a direct sum of the root subspaces of the operator $t\partial_t$. The V^\cdot -filtration and the canonical lattice	95
6.1	‘Block’ decomposition	95
6.2	Decomposition of a meromorphic connection \mathcal{M} into a direct sum of the root subspaces	96
6.3	The order function α and the V^\cdot -filtration	98
6.4	Identification of the zero fibre of the canonical extension \mathcal{L} and the canonical fibre of the fibration \underline{H}	99
6.5	The decomposition of sections $\omega \in \mathcal{M}$ into a sum of elementary sections	100
6.6	Transfer of automorphisms from the Milnor lattice H to the meromorphic connection \mathcal{M}	101
7	The limit Hodge filtration according to Varchenko and to Scherk–Steenbrink	103
7.1	Motivation of Scherk–Steenbrink’s construction of the Hodge filtration	103
7.2	The definition of the limit Hodge filtration F_{SS} according to Scherk–Steenbrink	106
7.3	The Scherk–Steenbrink theorem	108
7.4	Varchenko’s theorem about the operator of multiplication by f in Ω_f	110
7.5	The definition of the limit Hodge filtration F^\cdot on $H^n(X_\infty)$ according to Varchenko	111

	<i>Contents</i>	
		ix
	7.6 Comparison of the filtrations F_{SS} and F_{Va}	111
	7.7 Supplement on the connection between the Gauss– Manin differential system \mathcal{H}_X and its meromorphic connection \mathcal{M}	112
8	Spectrum of a hypersurface singularity	115
	8.1 The definition of the spectrum of an isolated singularity	115
	8.2 The spectral pairs $Spp(f)$	117
	8.3 Properties of the spectrum	118
	8.4 The spectra of a quasihomogeneous and a semi- quasihomogeneous singularity	119
	8.5 Calculation of the spectrum of an isolated singularity in terms of a Newton diagram	122
	8.6 Calculation of the geometric genus of a hypersurface singularity in terms of the spectrum	127
	8.7 Spectrum of the join of isolated singularities	127
	8.8 Spectra of simple, uni- and bimodal singularities	129
	8.9 Semicontinuity of the spectrum. Stability of spectrum for μ -const deformations	130
	8.10 Spectrum of a non-isolated singularity	132
	8.11 Relation between the spectrum of a singularity with a one-dimensional critical set and spectra of isolated singularities of its Iomdin series	134
III	The period map of a μ-const deformation of an isolated hypersurface singularity associated with Brieskorn lattices and MHSs	139
1	Gluing of Milnor fibrations and meromorphic connections of a μ -const deformation of a singularity	139
	1.1 Milnor fibrations	140
	1.2 Cohomological fibration	141
	1.3 Canonical extension of the sheaf \mathcal{H} and the meromorphic connection	142
2	Differentiation of geometric sections and their root components wrt a parameter	144
	2.1 Geometric sections and their root components	144
	2.2 Formulae for derivatives of geometric sections and their root components wrt a parameter	146
	2.3 Decomposition of the root components of geometric sections into Taylor series for upper diagonal deforma- tions of quasihomogeneous singularities	148
	2.4 The sheaves $Gr_V^\beta \mathcal{H}^{(0)}$	150

x	<i>Contents</i>	
3	The period map	151
	3.1 Identification of meromorphic connections in a μ -const family of singularities	151
	3.2 The period map defined by the embedding of Brieskorn lattices	152
	3.3 Example: the period map for E_{12} singularities	154
	3.4 The period map for hyperbolic singularities $T_{p,q,r}$	156
	3.5 The period map for simply-elliptic singularities	159
	3.6 The period map defined by MHS on the vanishing cohomology	163
4	The infinitesimal Torelli theorem	165
	4.1 The V -filtration on Jacobian algebra. The necessary condition for μ -const deformation	165
	4.2 Calculation of the tangent map of the period map. The horizontality of the MHS-period map	167
	4.3 The infinitesimal Torelli theorem	169
	4.4 The period map in the case of quasihomogeneous singularities	171
5	The Picard–Fuchs singularity and Hertling’s invariants	172
	5.1 The Picard–Fuchs singularity $PFS(f)$ according to Varchenko	172
	5.2 The Hertling invariant $Her_1(f)$	174
	5.3 The Hertling invariants $Her_2(f)$ and $Her_3(f)$	177
	5.4 Hertling’s results	179
	<i>References</i>	181
	<i>Index</i>	185