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132 Mixed Hodge Structures and Singularities

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Introduction

The aim of this book is to introduce and at the same time to survey some of the topics of singularity theory which study singularities by means of differential forms. Here differential forms associated with a singularity are the main subject as well as the main tool of investigation. Differential forms provide the main discrete invariants of a singularity as well as continuous invariants, i.e. they make it possible to study moduli of singularities of a given type.

A singularity is a local object. It is a germ of an algebraic variety, or an analytic space, or a holomorphic function. However, the majority of the ideas and methods, used in the theory under consideration, originated in the 'global' algebraic geometry. Therefore we first give a very brief and schematic description of these ideas.

The idea of using differential forms and their integrals to define numerical invariants of algebraic varieties goes back to the classic writers of algebraic geometry. It will be important for us that holomorphic and algebraic forms can be used to calculate the singular cohomology of a smooth algebraic variety over $\mathbb{C}$. Developing the ideas of Atiyah and Hodge (1955), Grothendieck (1966) showed that $H^i(X^{an}, \mathbb{C}) \simeq H^i_{\mathrm{DR}}(X/\mathbb{C})$, where $H^i_{\mathrm{DR}}(X/\mathbb{C})$ is the De Rham cohomology. Grothendieck defined $H^i_{\mathrm{DR}}(X/\mathbb{C})$ as the hypercohomology $\mathbb{H}^i(X, \Omega^{\cdot}_X)$ of the complex of sheaves of holomorphic differential forms on X. The comparison theorem enables us to calculate the cohomology of the complement to a hypersurface in projective space by means of the cohomology classes generated by rational differential forms.

Algebraic differential forms have also proved to be useful in the study of the monodromy of a family of complex varieties, using the Gauss–Manin connection. The monodromy transformation is the transformation of fibres (or their homotopic invariants) of a locally trivial fibration corresponding to a loop in the base. This notion appears when studying the multivalued analytic function, where it corresponds to the notion of the covering or sliding transformation. Very often the monodromy transformation appears in the following situation. Let $f: X \to S$ be a proper holomorphic map of an analytic space to the disk in the complex plane. Let X_t be the fibre $f^{-1}(t)$, $t \in S$, $S' = S \setminus \{0\}$ and $X' = f^{-1}(S')$. By reducing the radius of S, if necessary, we can make the fibration $f': X' \to S'$ a locally trivial C^∞-fibration. The monodromy transformation T associated with the loop in S' surrounding 0 is called the monodromy transformation of the family f. The action of the monodromy T on the vector space $H^*(X_t)$ is obtained by the parallel displacement of the cohomology classes in the fibres of the locally constant fibration $\underline{H} = \bigcup_{t \in S'} H^i(X_t, \mathbb{C}) = R^i f_* \mathbb{C}_{X'}$. Grothendieck also defined the relative De Rham cohomology sheaves $\mathscr{H}^i_{\mathrm{DR}}(X/S) \simeq \mathbb{R}^i f_*(\Omega^\cdot_{X/S})$. If $f: X \to S$ is a smooth proper morphism of algebraic varieties/$\mathbb{C}$, then from Grothendieck's theorem it follows the existence of the canonical isomorphism of coherent analytic sheaves $\mathscr{H}^i_{\mathrm{DR}}(X/S)^{\mathrm{an}} \simeq R^i f_*{}^{\mathrm{an}}(\mathbb{C}) \otimes_{\mathbb{C}} \mathscr{O}_{S^{\mathrm{an}}}$. The presence of the locally constant sheaf $\underline{H} = R^i f_*(\mathbb{C})$ in $\mathscr{H}^i_{\mathrm{DR}}(X/S)$ defines a topological connection on the sheaf $\mathscr{H}^i_{\mathrm{DR}}(X/S)$. Katz and Oda (1968) gave an algebraic definition of the canonical connection (the Gauss–Manin connection) on the sheaves $\mathscr{H}^i_{\mathrm{DR}}(X/S)$ such that the sheaves of its horizontal sections are $R^i f_*(\mathbb{C})$. They calculated this connection explicitly and showed that it reduces to the definition orginally given by Manin (1958) for the case in which X/S is an algebraic curve over the field of functions.

The family of varieties X_t defined by the morphism f degenerates at the point $0 \in S$, and the Gauss–Manin connection has a singularity at the point 0. This singularity is regular. The notion of a regular connection generalizes the classical notion of a differential equation with a regular singular point, and is the subject of Deligne's book [D1]. Katz gave an algebraic proof of the regularity of the Gauss–Manin connection (1970). Analytic proofs were given by Griffiths (1971) and by Deligne [D1]. The regularity theorem is related to the monodromy theorem. When the space X as well as all fibres X_t, $t \neq 0$, is smooth, the monodromy theorem states that T is quasi-unipotent on $H^*(X_t, \mathbb{Q})$, i.e. there exist positive integers k and N such that $(T^k - 1)^N = 0$. There are several proofs of this basic theorem (Clemens 1969; Katz 1971; Griffiths & Schmid 1975). Many of the characteristic features of the degenerate family $f: X \to S$ become apparent in the properties of the monodromy. The monodromy of the family f is closely connected with the mixed Hodge structure (MHS) on the cohomology

$H^*(X_0)$ and $H^*(X_t)$. The Hodge structure and the period map of a family of algebraic varieties defined by the Hodge structure on the cohomology of fibres $H^*(X_t)$ are the second most important notions used in this book.

The concept of a pure Hodge structure is a formalization of the structure of the cohomology groups of a compact Kähler manifold. From the theory developed by Hodge in the 1930s it follows that $H^n(X, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}$, $\overline{H^{p,q}} = H^{q,p}$, where the cohomology vector space is identified with the vector space of harmonic forms, and harmonic forms, and consequently the cohomology, can be decomposed into the direct sum of components of the type (p, q). Under a variation of a variety X in a family X_t the variation of subspaces $H^{p,q}(X_t)$ is not complex analytic. For this reason it is more convenient to study the Hodge filtration $F^p H^n(X, \mathbb{C}) = \oplus_{r \geqslant p} H^{r,n-r}$, which does depend analytically on the parameter t. One can give an equivalent definition of the Hodge structure in terms of the Hodge filtration. Using the isomorphism $H^n(X, \mathbb{C}) \simeq H^n_{\mathrm{DR}}(X)$, we can express the Hodge filtration in terms of the stupid (obvious) filtration $\sigma_{\geqslant p}\Omega^{\cdot}_X = \{0 \to \Omega^p_X \to \Omega^{p+1}_X \to \ldots\}$ on the De Rham complex. One of Griffiths's discoveries [Gr] was that the Hodge filtration is related to the pole order filtration: if ω has a pole of order not greater than $k+1$ along a non-singular projective variety X, then the residue $\operatorname{Res}\omega$ is a sum of forms of the type (p, q) with $q \leqslant k$. This gives a purely algebraic definition of the Hodge filtration.

The theory of MHS developed by Deligne [D2, D3] is used more and more. The definition of the MHS includes, besides a decreasing Hodge filtration $F^{\cdot}$, an increasing weight filtration W. Deligne showed that the cohomology of any algebraic variety (possibly non-proper and singular) has a natural MHS. The MHS also appears in investigations of degenerations of algebraic varieties. Let a morphism $f\colon X \to S$ define a family of non-singular projective varieties over the punctured disk S', and let $X_0 = f^{-1}(0)$ be the degenerate fibre. Schmid [Sm] and Steenbrink [S1] investigated the question of what happens with the Hodge structure on $H^n(X_t, \mathbb{Z})$, when t is limited to the point 0. The limit object appears to be a MHS. The Hodge filtration of the limit MHS is in a sense the limit of the Hodge filtration on $H^n(X_t)$, and the weight filtration is related to the monodromy.

The study of period maps goes back to the investigations of Abel and Jacobi on integrals of algebraic functions. A tempestuous development of the theory of periods of integrals begins after Griffiths's papers (1968). Griffiths studied the properties of integrals in terms of the notions of the period matrix space and the period map which he introduced. Let us consider a family X_t of non-singular projective varieties depending on a parameter $t \in S$, and defined by a proper morphism $f\colon X \to S$. Using the connection on the fibration $\underline{H} = \bigcup_{t \in S} H^n(X_t)$, we can displace the Hodge structures on $H^n(X_t)$ to the cohomology space $H = H^n\ (X_{t_0})$ of a fixed fibre. Considering the Hodge filtration $F^p H^n(X_t)$ only, we can associate a flag $F^{\cdot}(t)$ in H to every point $t \in S$,

$$H = F^0(t) \supset F^1(t) \supset \ldots \supset F^{n+1}(t) = \{0\},$$

and hence obtain a point $F^{\cdot}(t)$ of the manifold of flags of given type. We obtain the period map $\Phi\colon S \to \mathscr{F}$. In coordinates the period map is given by the periods of integrals (we have to choose a basis in the homology space consisting of continuous families of cycles, and to define the subspaces $F^p(t)$ by bases of differential forms). In fact the definition of Φ is not correct, since as t moves round a loop in S, the identification of $H^n(X_{t_0})$ with itself need not be the identity. So we have to consider either a map $\tilde{\Phi}\colon \tilde{S} \to \mathscr{F}$ of the universal cover of S, or a map Φ of S to a quotient of $\mathscr{F}$ by some group.

Griffiths found that to first order, F^p is deformed only into the subspace F^{p-1}. In terms of the Gauss–Manin connection this can be interpreted as $\nabla F^p \subset \Omega^1_S \otimes F^{p-1}$ (the horizontality theorem). In fact, Griffiths considered polarized Hodge structures on the primitive cohomology groups $P^n(X_t)$. He constructed the period matrix space D of all possible polarized Hodge structures of given type, which is a submanifold of the flag manifold distinguished by the Hodge–Riemann bilinear relations. It turns out that D is an open homogeneous complex manifold, there is a naturally defined properly discontinuous group Γ of analytic automorphisms of D such that $M = D/\Gamma$ is an analytic space, and then we obtain the period map $\Phi\colon S \to M$.

The period map can be used for the description of the moduli of algebraic varieties. Here there are problems about Torelli-type theorems. The global Torelli problem is the question of whether the period map $\Phi\colon \mathscr{M} \to M$ of the moduli space of algebraic varieties of a given type is an embedding, i.e. whether the period matrix uniquely characterizes the polarized algebraic variety. The affirmative answer to this question in the case of algebraic curves is the usual Torelli theorem. The local Torelli problem is one of deciding when the Hodge structure on $H^*(X_t, \mathbb{C})$ separates points in the local moduli space (Kuranishi space) of X_t. The infinitesimal Torelli problem is one of deciding when the tangent map $d\Phi$ to the period map of the universal family is a monomorphism. The criterion for the infinitesimal Torelli theorem to hold, which was obtained by

Griffiths (1968), stimulated the appearance of many papers on this theme. We are not able in this introduction to go in detail into the problems touched upon above, and we refer the reader to one of the surveys on the theory of Hodge structures and periods of integrals, e.g. to [K-Ku] and [B-Z], where one can find other references.

The aim of this book is to transfer the ideas and notions, described above, to the local situation – to the case of isolated singularities of holomorphic functions $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Again we have a morphism $f: X \to S$, but now the fibres are local analytic hypersurfaces in an open set $X \subset \mathbb{C}^{n+1}$. In Chapter I we introduce the main personage of this book – the meromorphic Gauss–Manin connection $\mathscr{M}$ of a singularity f and the Brieskorn lattice $\mathscr{H}^{(0)}$ in it. We prove the regularity of the singularity of the Gauss–Manin connection and the monodromy theorem. The discussion in Chapter I is based on the classical papers of Brieskorn [Br] and Malgrange [M]. In Chapter II we consider the limit MHS appearing on the vanishing cohomology $H^n(X_t, \mathbb{C})$ of an isolated singularity f. In the main we follow the development as it occurred historically. Here the main contribution comes from the papers of Steenbrink, Varchenko and Scherk. Initially the MHS on the vanishing cohomology $H^n(X_t)$ was constructed by Steenbrink [S3] following a suggestion of Delinge. He used an embedding of the morphism $f: X \to S$ in a projective family $Y \to S$ and the limit MHS in the case of a degeneration of projective varieties [S1]. Then Varchenko [V2, V3] proposed and accomplished the direct introduction of the limit Hodge filtration $F^\cdot$ on $H^n(X_\infty, \mathbb{C})$ (by means of asymptotics of integrals), without using an embedding to a projective family. Following this idea Scherk and Steenbrink [Sc-S] introduced the filtration $F^\cdot$ in a different way. They showed how the filtration $F^\cdot$ is obtained from the embedding of the Brieskorn lattice $\mathscr{H}^{(0)}$ in the meromorphic connection. Finally, in Chapter III we consider the period map of a μ-const deformation $f_y(x)$ of isolated singularities parametrized by points $y \in Y$ of a non-singular manifold. The basis of this chapter is the papers of Saito [Sa8], Karpishpan [Ka2] and Hertling [He1, He2]. First we consider the period map defined via embedding of Brieskorn lattices but not the one defined via the limit MHS on $H^n(X_t, \mathbb{C})$ as it should be according to the ideology presented at the beginning of this introduction. Instead it is connected with the following. Firstly, the limit MHS is determined by the embedding of Brieskorn lattice, and, secondly, the limit MHS on $H^n(X_t, \mathbb{C})$ is a rougher invariant than the embedding $\mathscr{H}^{(0)} \subset \mathscr{M}$ is (because the filtration $F^\cdot$ is defined not by the embedding $\mathscr{H}^{(0)} \subset \mathscr{M}$ but by the embedding of adjoint objects $Gr^\cdot_V\mathscr{H}^{(0)} \subset Gr^\cdot_V\mathscr{M}$). We give examples of the explicit calculation of the period maps for the deformations of unimodular singularities and prove the horizontality theorem and the infinitesimal Torelli theorem.

Now we give a detailed description of the contents of this book. In §1 of Chapter I we recall the definition of the Milnor fibration [Mi] $f\colon X \to S$ of an isolated singularity $f\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. The restriction of f over the punctured disk S' is a smooth locally trivial fibration. We introduce the cohomological fibration $\underline{H} = \bigcup_{t\in S'} H^n(X_t, \mathbb{C})$ defining the topological Gauss–Manin connection ∇_{top} on the sheaf of sections $\mathscr{H} = \underline{H} \otimes \mathscr{O}_{S'} = R^n f_* \mathbb{C}_{X'} \otimes \mathscr{O}_{S'}$. In §2 we develop elements of the theory of connections on locally free sheaves, the presentation of which one can find in Deligne's book [D1]. We pay special attention to the relation between ∇ and the dual to it connection ∇^*. For Gauss–Manin connections, determined by the homological and cohomological fibrations, this leads to the period matrix of a singularity f being a solution of system of linear differential equations – the Picard–Fuchs equation of the singularity.

In §3 we introduce the De Rham cohomology sheaf $\mathscr{H}^n_{\text{DR}}(X/S) = \mathscr{H}^n(f_*\Omega^\cdot_{X/S})$, which is a natural extension of the sheaf $\mathscr{H}$ to the whole disk. This sheaf establishes the connection between topology and algebra and analysis, and reflects the analytic nature of the singularity f. §4 contains Brieskorn's calculation of the Gauss–Manin connection of the sheaf $\mathscr{H}^{(-2)} = \mathscr{H}^n_{\text{DR}}(X/S)$ in terms of differential forms. This calculation naturally leads to the sheaf $\mathscr{H}^{(-1)} = f_*\Omega^n_{X/S}/d(f_*\Omega^{n-1}_{X/S})$ which contains $\mathscr{H}^{(-2)}$ and is also an extension of the sheaf $\mathscr{H}$. We introduce the notion of the meromorphic Gauss–Manin connection on the sheaf $\mathscr{M} = \mathscr{H}^{(-2)} \otimes \mathscr{O}_S[t^{-1}]$. We also explain how to give a more conceptual description of the calculation of the Gauss–Manin connection on $\mathscr{H}^{(-2)}$ as a connecting homomorphism in an exact cohomology sequence. In §5 the main personage of this survey appears. This is the Brieskorn lattice $\mathscr{H}^{(0)}$, the third natural extension of the sheaf $\mathscr{H}$ to the whole disk S. The lattice $\mathscr{H}^{(0)}$ is defined in terms of $(n+1)$-forms. The identification of $\mathscr{H}^{(0)}$ and $\mathscr{H}^{(-1)}$ on S' is realized by means of the Leray derivative (the Poincaré residue), $\omega \mapsto \omega/\mathrm{d}f = \mathrm{Res}\,[\omega/(f-t)]$. All three lattices $\mathscr{H}^{(-i)}$, $i = 0, 1, 2$, are the terms of an increasing filtration on $\mathscr{M}$. The correlation between them is contained in the diagram shown in (I.5.3.4). In §6 from Malgrange's theorem, which claims that the periods of an n-form ω have the limit $\lim_{t\to 0} \int_{\gamma(t)} \omega = 0$, we obtain a result of Sebastiani about the absence of torsion in the sheaves $\mathscr{H}^{(-i)}$, and hence we obtain that these sheaves are locally free $\mathscr{O}_S$-modules of rank μ.

In §7 we recall the classical definition of a regular singular point of a system of linear differential equations and Sauvage's theorem on the

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regularity of a simple singular point. We give the definition of a connection with regular singularity and of the residue of such a connection wrt a saturated lattice. In §8 we prove a fundamental theorem on the regularity of the Gauss–Manin connection. We give two proofs of this important fact. Firstly, we prove that periods of holomorphic forms give solutions of the Picard–Fuchs equation. Then from Malgrange's theorem it follows that horizontal sections of the Gauss–Manin connection have a moderate growth. Secondly, using a resolution of the singularity and the sheaves of differential forms with logarithmic poles, we construct a saturated lattice in the meromorphic Gauss–Manin connection. In §9 from the regularity of the Gauss–Manin connection we deduce the monodromy theorem claiming that all eigenvalues of the monodromy are roots of unity. We give a beautiful proof of this due to Brieskorn, based on the positive solution of the seventh Hilbert problem. The part of the monodromy theorem concerning the size of Jordan blocks is proved in (II.3.5.9). We obtain corollaries of the monodromy theorem about the decomposition into series of the periods of integrals of differential forms.

In §10 we consider non-isolated hypersurface singularities. Starting from the construction of the Gauss–Manin connection as a connecting homomorphism in an exact sequence of complexes and in the spirit of [Sr], we obtain a natural generalization of Brieskorn lattices $\mathscr{H}^{(i)}$ to the case of non-isolated singularities. We give Van Straten's criterion on the absence of torsion in the sheaves $\mathscr{H}^{(i)}$ and its application to the non-isolated singularities with a one-dimensional critical set.

In Chapter II we consider the limit MHS appearing on the vanishing cohomology $H^n(X_t, \mathbb{C})$ of an isolated hypersurface singularity. In §1 we recall very briefly the necessary basic definitions from the theory of MHS developed by Deligne. In §2 we introduce the limit MHS constructed by Schmid [Sm] for a variation $(\underline{H}, \mathscr{F}^{\cdot})$ of pure Hodge structures $F_t^{\cdot}$ on a vector space H, parametrized by points of the punctured disk S'. The limit Hodge filtration $F^{\cdot}$ appears on the zero fibre $\mathscr{L}/t\mathscr{L}$ of the canonical extension $\mathscr{L}$ of the sheaf $\mathscr{H} = \underline{H} \otimes \mathscr{O}_{S'}$ to the point 0. The weight filtration $W_{\cdot}$ is the weight filtration $W(N)$ of the nilpotent operator $N = -(1/2\pi i)\log T_u$. In §3 we introduce the limit MHS on the vanishing cohomology $H = H^n(X_\infty, \mathbb{C})$ (X_∞ is the canonical fibre of the Milnor fibration) according to Steenbrink. Steenbrink used an embedding of the Milnor fibration $X \to S$ to a projective family $Y \to S$. He introduced the limit MHS on $H^n(Y_\infty, \mathbb{C})$ using a resolution of singularities of the zero fibre and the complex of relative differential forms with logarithmic poles. We give Steenbrink's construction only very schematically, without going into technical details. The limit MHS on $H^n(X_\infty, \mathbb{C})$ constructed by Steenbrink can be considered as a quotient of the MHS on $H^n(Y_\infty, \mathbb{C})$. From here it follows the symmetry of Hodge numbers of the MHS on $H^n(X_\infty, \mathbb{C})$, and also the monodromy theorem on the size of Jordan blocks.

In §4 we consider the Hodge theory of a non-singular hypersurface Y in a non-singular manifold X developed by Griffiths and extended by Deligne to the case of divisors with normal crossings. This theory enables us to calculate the cohomology of Y and of the complement $X \setminus Y$ by means of differential forms on X with poles on Y, and also relates the Hodge filtration $F^\cdot$ to the pole order filtration $P^\cdot$. When Y has singularities, the filtrations $F^\cdot$ and $P^\cdot$ do not, generally, coincide [Ka1, D-Di]. We apply this theory for the description of $H^n(X_t)$, where the hypersurface $X_t \subset X$ is the fibre of the Milnor fibration. The relative variant of this theory enables us to obtain a natural extension $\mathscr{H}_X$ of the sheaf $\mathscr{H}$ to S. Then $\mathscr{H}_X$ is described as the cohomology sheaf $\mathscr{H}_X = \mathscr{H}^{n+1}(K^\cdot, \underline{d})$ of the bicomplex $(K^{\cdot\cdot}, \mathrm{d}_1, \mathrm{d}_2)$, the terms of which are differential forms with poles, and the degrees are defined by the order of a differential form and the order of its pole. In the bicomplex $K^{\cdot\cdot}$ the Poincaré complex and the Koszul complex are intertwined, and its differentials are exterior differentiation $\mathrm{d}_1 = \mathrm{d}$ of differential forms and exterior multiplication by the form df. The Gauss–Manin connection and the Hodge filtration $F^\cdot$ on $\mathscr{H}_X$ are defined in terms of this bicomplex. In fact $\mathscr{H}_X$ is a D-module, or a differential system, which appears in the papers of Pham [Ph1, Ph2], Scherk and Steenbrink [Sc-S]. The study of this is extended in the papers of M. Saito [Sa1–Sa10] in the frames of the theory of D-modules and of the 'monstrous' theory of mixed Hodge modules developed by him. This latter theory raises the level of abstractness by at least one more order, and moves further from the soil, in which all this began to grow. The 'new' homological algebra, derived cathegories, perverse sheaves, the Riemann–Hilbert correspondence, etc. are essentially used in it. Under Saito's influence the theory becomes more and more abstract and technical. One of the aims of this book was not to follow this line but to evolve the theory using the 'traditional' (i.e. habitual to algebraic geometricians) language of sheaves, connections, spectral sequences etc. and to avoid using the language of the theory of D-modules, the theory of mixed Hodge modules etc. This is possible because of the following circumstance. As is shown in §5 the operator ∂_t is invertibl $\mathscr{H}_X$, $\mathscr{H}_X$ contains the Brieskorn lattice $\mathscr{H}^{(0)} = F^n\mathscr{H}_X$ as well as canonical lattice $\mathscr{L}$, $\mathscr{H}^{(0)} \subset \mathscr{L} \subset \mathscr{H}_X$. That is the pair $\mathscr{H}^{(0)} \subset \mathscr{L}$ tains all the information which is of interest to us. The localization of differential system $\mathscr{H}_X$ leads us to the classical frames of the meromorphic connection $\mathscr{M} = (\mathscr{H}_X)_{(t)}$, and as is explained in the supplement to §7, the pair $\mathscr{H}^{(0)} \subset \mathscr{L}$ happily stands localization and removes to $\mathscr{M}$, $\mathscr{H}^{(0)} \subset \mathscr{L} \subset \mathscr{M}$. All this enables us to work in the classical frames of the meromophic connection $\mathscr{M}$.

In §6 we study the structure of the meromorphic connection $\mathscr{M}$, its decomposition into the sum $\mathscr{M} = \hat{\oplus}_\alpha C_\alpha$ of root subspaces of the operator $t\partial_t$, the $V^\cdot$-filtration on $\mathscr{M}$. We determine the isomorphism ψ: $H \rightrightarrows \mathscr{L}/t\mathscr{L} = \oplus_{-1<\alpha\leqslant 0} C_\alpha$ between the canonical fibre of the fibration $\underline{H}$ and the zero fibre of the canonical lattice $\mathscr{L} = V^{>-1}\mathscr{M}$, and it enables us to introduce the MHS on H in terms of the lattice $\mathscr{L}$.

In §7 we introduce the MHS on $H = H^n(X_\infty, \mathbb{C})$, according to Varchenko [V3] and Scherk and Steenbrink [Sc-S]. At first we define the limit Hodge filtration $F^\cdot$ according to Scherk and Steenbrink, who transformed the approach of Varchenko and gave the definition of $F^\cdot$ in terms of the differential system $\mathscr{H}_X$ and the embedding $\mathscr{H}^{(0)} \subset \mathscr{L}$, and then we define $F^\cdot$ according to Varchenko. Such an order of exposition is connected with the fact that, in our opinion, the construction of Scherk and Steenbrink has genetically a more natural motivation. In the beginning we observe the sequence of steps leading to the construction of $F^\cdot$ by Scherk and Steenbrink.

In §8 we study the main discrete invariant of a hypersurface singularity – its spectrum Sp(f), and a more detailed invariant Spp(f) the set of spectral pairs. To give Spp(f) is equivalent to giving Hodge numbers $H_\lambda^{p,q}$. The spectrum Sp(f) is a set of μ rational numbers $\alpha_1, \dots, \alpha_\mu$, where $\alpha_j = -(1/2\pi i)\lambda_j$ are logarithms of the eigenvalues λ_j of the monodromy T. It codes the relation between the semisimple part of the monodromy and the limit Hodge filtration $F^\cdot$. We study the properties of spectrum. We show that $Sp(f) \subset (-1, n)$ and that the spectrum is symmetric wrt the centre of this interval $(n-1)/2$. We develop the techniques for the calculation of the spectrum. We explain: how to find the spectrum of a (semi)quasihomogeneous singularity; how to find the spectrum in terms of the Newton filtration defined by the Newton boundary; and in particular, how to find the negative part of spectrum, the degree of which, $\sum_{-1<\alpha\leqslant 0} n_\alpha = p_g$, is equal to the geometric genus of the singularity [S3]; how to find the spectrum of the join of isolated singularities, and in particular, how the spectrum changes under adding squares of new variables. This technique enables us to find spectra of all the simple, uni- and bi-modal singularities, which we gather together in a table. Finally, we study variations of MHS of families of hypersurface singularities. We deal with this more widely in Chapter III, but in §8 we consider the behavior of discrete invariants under deformations. We give the results of Varchenko and Steenbrink on the semicontinuity of spectrum and on its stability under μ-const deformations. At the end of §8, following Steenbrink, we define the spectrum of non-isolated singularities and give a theorem on the relation of the spectrum of a singularity with a one-dimensional critical set and the spectra of isolated singularities of its Iomdin series.

In §1 of Chapter III we begin to study μ-const deformations $f_y(x)$ of isolated singularities parametrized by the points $y \in Y$ of a non-singular variety. We explain how to glue the objects, associated earlier with an 'individual' singularity, to a family parametrized by Y. In particular, we obtain the family of Milnor fibrations $X(y)$, the family of cohomological fibrations $\underline{H} = \bigcup \underline{H}(y)$, the family of meromorphic connections $\mathscr{M} = \hat{\oplus}_\beta C_\beta$, etc. In §2 we obtain the formula for differentiating wrt parameter y geometric sections $s[\omega](t, y)$, defined by a holomorphic $(n+1)$-form $\omega = g(x, y)\,dx$. The same formula is used for differentiating the root components $s(\omega, \beta)$ of the geometric sections $s[\omega](y) = \sum_{\beta > -1} s(\omega, \beta)(y)$. In the case of upper diagonal deformations of a quasi-homogeneous singularity $f(x)$ from this formula we obtain a formula for decomposition of $s(\omega, \beta)(y)$ into Taylor series in degrees of y. In §3 first we define the period map $\Phi\colon\ Y \to \Pi$ defined by the embedding of Brieskorn lattices. For all Brieskorn lattices $\mathscr{H}^{(0)}(y)$ we have inclusions $V^{>-1} \supset \mathscr{H}^{(0)}(y) \supset V^{n-1}$. The period map takes a point y to the subspace $\mathscr{H}^{(0)}(y) \bmod V^{n-1}$ in the finite-dimensional vector space $V^{>-1}/V^{n-1}$, i.e. to the point in the Grassman manifold Π. Then we define the period map $\overline{\Phi}$ defined by the MHS on the vanishing cohomology. This period map takes a point y to the Hodge filtration $F^{\cdot}(y)$ of the limit MHS, i.e. to the point in the flag manifold. We give, following Hertling, explicit calculations of the period maps of universal families of unimodular singularities. In §4 we calculate the tangent map of the period map and prove the horizontality theorem and the infinitesimal Torelli theorem for the period map Φ. We compare the period maps Φ and $\overline{\Phi}$ of the miniversal μ-const deformation of a quasihomogeneous singularity. From this comparison it follows that the Torelli theorem for the period map $\overline{\Phi}$ in general is false. Finally, in §5 we consider the 'global' Torelli problem. Varchenko introduced the notion of the Picard–Fuchs singularity PFS(f) of a singularity f in terms of framed Picard–Fuchs equations. Hertling reformulated this in terms of embeddings of Brieskorn lattices. We interpret it as a point in the quotient Π/G_1 of the period space by the group $G_1 \subset GL(H_{\mathbb{Z}})$ of automorphisms commuting with the monodromy. Then the Torelli problem for the family f_y, $y \in Y$, concerns the injectivity of the map Φ_1: $Y/\sim \to \Pi/G_1$, where '$\sim$' is the equivalence relation induced by the R-equivalence of singularities. We give Hertling's results [He1, He2].

The reader is expected to have the knowledge and training usual in algebraic and analytic geometry. This includes knowledge of sheaf theory and the technique of spectral sequences.

The list of references reflects the development of singularity theory. It includes: firstly, the original papers of E. Brieskorn, B. Malgrange, J.H.M. Steenbrink, J. Scherk, A.N. Varchenko, F. Fham, M. Saito, Ya. Karpishpan and C. Hertling, directly connected with the considered topic; secondly, some general papers on MHS and periods of integrals (P. Griffiths, P. Deligne, W. Schmid); thirdly, some books and surveys [Mi, AGV, Ph1, D1, Di3, B-Z]. Some other references are also included.

Now about numbering and cross-references in this book. Each of three chapters is divided into sections, and each section is divided into subsections. In a subsection all claims, remarks and displayed formulae are numerated successively in a uniform way by three numbers, the first of which is the number of the section and the second the number of the subsection. For example, the tag (1.3.2) means claim (or formula) 2 in subsection 3 of section 1. To refer to a claim etc. we use three numbers within a given chapter, and four numbers, the first of which is a roman numeral, to refer to other chapters. Thus (II.7.5.2) refers the reader to claim 2 of subsection 5 of section 7 of chapter II.

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