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Excerpt
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The Gauss–Manin connection

1 Milnor fibration, Picard–Lefschetz monodromy transformation, topological Gauss–Manin connection

1.1 Milnor fibration

(1.1.1) Let $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of holomorphic function $t = f(x_0, x_1, \ldots, x_n)$. We assume as a rule that f has an isolated critical point at $0 \in \mathbb{C}^{n+1}$, in other words, f has (or is) an isolated singularity. Let $B = \{|x| < \varepsilon\} \subset \mathbb{C}^{n+1}$ be a ball of radius ε, and let $S = \{|t| < \delta\} \subset \mathbb{C}$ be a disk of radius δ. Put $X = B \cap f^{-1}(S)$ and let f also denote the restriction of f onto X, $f: X \to S$. Let $S' = S \backslash \{0\}$ be the punctured disk, $X_t = f^{-1}(t)$ be the fibre over the point $t \in S'$, and $X_0 = f^{-1}(0)$ be a singular fibre, $X' = X \backslash X_0$. Denote by f' the restriction of f on X'

$$\begin{array}{ccccc} X & \supset & X' & \supset & X_t \\ f\downarrow & & \downarrow f' & & \\ S & \supset & S' & \ni & t. \end{array}$$

As is shown by Milnor [Mi], if ε and $\delta \ll \varepsilon$ are sufficiently small, then f' is a smooth locally trivial fibration, the diffeomorphism type of which only depends on the germ of f at 0. Usually the fibration $f': X' \to S'$ is called the Milnor fibration. In the following it will be convenient for us to call the whole morphism $f: X \to S$ the *Milnor fibration*. Any fibre X_t, $t \in S'$, is called a *Milnor fibre*. We can think of the singular fibre X_0 as a degeneration of a family of manifolds X_t. Replacing the fibres X_t by their (co)homology we get a 'linearization' of the family X_t.

1.2 Cohomological Milnor fibration

(1.2.1) A Milnor fibration f' defines a vector bundle $\underline{H} \to S'$ on S', or a locally constant sheaf (or a system of local coefficients, in different terminology)

$$\underline{H} = \bigcup_{t \in S'} H^p(X_t, \mathbb{C}) = R^p f_* \mathbb{C}_{X'}$$

and it also defines a dual vector bundle

$$\underline{H}_* = \mathrm{Hom}_{\mathbb{C}_{X'}}(\underline{H}, \mathbb{C}_{S'}) = \bigcup_{t \in S'} H_p(X_t, \mathbb{C}).$$

We call $\underline{H}$ the *cohomological*, and $\underline{H}_*$ the *homological Milnor fibration* of a singularity f.

1.3 Topological Gauss–Manin connection

(1.3.1) Denote by

$$\mathscr{H} = \underline{H} \underset{\mathbb{C}_{S'}}{\otimes} \mathscr{O}_{S'} = R^p f_* \mathbb{C}_{X'} \underset{\mathbb{C}_{S'}}{\otimes} \mathscr{O}_{S'}$$

the locally free sheaf of sections of the fibration $\underline{H}$. A local section w of $\underline{H}$ can be viewed as a family of cohomology classes $w(t)$ depending on parameter t. The flat system $\underline{H}$ in the sheaf $\mathscr{H}$ allows us to translate cohomology classes $w(t)$ from a fibre X_t to a nearby fibre and, consequently, allows us to differentiate the cohomology classes wrt parameter, and to define a connection on the sheaf (see the §2). This connection is called the *topological* (or transcendental) *Gauss–Manin connection* and is denoted by Δ_{top}.

1.4 Picard–Lefschetz monodromy transformation

(1.4.1) A locally constant bundle $\underline{H}$ (or $\underline{H}_*$) defines (and is defined by) an action of the fundamental group $\pi_1(S', t)$ on a fibre $\underline{H}_t = H^p(X_t, \mathbb{C})$. Let $\gamma\colon [0, 1] \to S'$ be a loop representing an element of $\pi_1(S', t)$. The inverse image of the locally trivial fibration $f'\colon X' \to S'$ defines a locally trivial, and consequently, trivial fibration $\gamma^{-1}X' \xrightarrow{\varphi} [0, 1]$ on the segment. A trivialization of this family defines a diffeomorphism $h_\gamma\colon X_t = \varphi^{-1}(0) \to X_t = \varphi^{-1}(1)$. We thus obtain the monodromy representation

$$\pi_1(S', t) \to \mathrm{Aut}\, H^p(X_t, \mathbb{C}), \quad [\gamma] \mapsto (h_\gamma^*)^{-1}. \tag{1.4.2}$$

Let $[\gamma] \in \pi_1(S', t)$ be the generator of the fundamental group represented by a counter-clockwise oriented circle γ around the origin. The linear transformations

$$M = (h_\gamma)_*\colon H_p(X_t, \mathbb{C}) \to H_p(X_t, \mathbb{C}) \tag{1.4.3}$$

$$T = (h_\gamma^*)^{-1}\colon H^p(X_t, \mathbb{C}) \to H^p(X_t, \mathbb{C}) \tag{1.4.4}$$

are called the (local) *Picard–Lefschetz monodromy* transformation of homology and cohomology, respectively.

(1.4.5) The Milnor fibration, the Gauss–Manin connection and the Picard–Lefschetz monodromy are the main objects associated with a singularity f and they contain a great deal of information about this singularity. These objects can be studied by topological methods. In the case of isolated singularities the most important fact ([Mi]) is that the fibre X_t has the homotopy type of a bouquet $S^n \vee \cdots \vee S^n$ of μ spheres of dimension n. In particular, the Betti numbers $b_i = \dim H^i(X_t, \mathbb{C}) = 0$ for $i \neq 0$ and n and, consequently, $\underline{H} = R^n f_* \mathbb{C}_{X'}$ is the only non-trivial fibration among the fibrations $R^p f_* \mathbb{C}_{X'}$. The number $\mu = \dim H^n(X_t, \mathbb{C}) = b_n$ (the rank of this fibration) is called the *Milnor number* of the singularity f. We refer the reader to other surveys ([AGV], ch. 1; [Di3]) for topological methods for the study of singularities. In this survey we'll be interested in the study of singularities by algebraic methods in which the main subject as well as the main tool of investigation is the differential forms associated with a singularity.

2 Connections, locally constant sheaves and systems of linear differential equations

2.1 Connection as a covariant differentiation

Let S be a complex manifold of dimension m, and $\mathscr{E}$ be a locally free sheaf of rank n on S. The presence of a connection on $\mathscr{E}$ enables us to differentiate sections of $\mathscr{E}$ along vector fields on the base S. On the other hand, the notion of a connection on $\mathscr{E}$ is a way of having an invariant definition of S of a system of homogeneous linear differential equations with n unknown functions in m variables. In the following we'll be mainly interested in the Gauss–Manin connection. First we briefly recall the necessary facts about connections. An excellent exposition of this subject can found in Deligne's paper [D1].

(2.1.1) *Definition* A *connection* on a quasicoherent sheaf of $\mathcal{O}_S$-modules $\mathscr{E}$ is a $\mathbb{C}$-linear homomorphism

$$\nabla\colon \mathscr{E} \to \Omega^1_S \underset{\mathcal{O}_S}{\otimes} \mathscr{E} = \Omega^1_S(\mathscr{E})$$

satisfying the *Leibniz identity*

$$\nabla(gs) = dg \otimes s + g\nabla s,$$

where g and s are local sections of sheaves $\mathcal{O}_S$ and $\mathcal{E}$, respectively. ∇ is called the *covariant differentiation*.

This notion originates from the corresponding classical definition. The classical definition of a connection on $\mathcal{E}$ enables us to compare infinitely close fibres of the sheaf $\mathcal{E}$ and consists in giving for any pair of points x and y, infinitely close to first order in S, an isomorphism of fibres $\gamma: \mathcal{E}(x) \to \mathcal{E}(y)$. To translate this into modern language we have to consider points in the categorical sense, i.e. morphisms $T \to S$. Then two T-valued points $x: T \to S$ and $y: T \to S$ are said to be infinitely close to first order, if $(x, y): T \to S \times S$ factors through the neighbourhood of the first order $\Delta_1 = (\Delta, \mathcal{O}_{X\times X}/I^2)$ of the diagonal Δ in $S \times S$. In this way we come naturally to the previous formal definition [D1].

2.2 Equivalent definition: a covariant derivative along a vector field

(2.2.1) For a local vector field w on S, i.e. a local section of the sheaf of vector fields $\theta_S = \mathcal{H}om_{\mathcal{O}_S}(\Omega^1_S, \mathcal{O}_S)$, and for a section s of the sheaf $\mathcal{E}$ the *covariant derivative* of s along w is defined by

$$\nabla_w(s) = \langle \nabla s, w \rangle \tag{2.2.2}$$

where $\langle , \rangle$ is induced by the pairing $\Omega^1_S \times \theta_S \to \mathcal{O}_S$.

∇_w defines a $\mathbb{C}$-linear homomorphism

$$\nabla_w: \mathcal{E} \to \mathcal{E} \tag{2.2.3}$$

and the Leibniz identity transforms into the (Leibniz) identity

$$\nabla_w(gs) = w(g)s + g\nabla_w(s). \tag{2.2.4}$$

We obtain one more definition of the notion of connection on $\mathcal{E}$ as an $\mathcal{O}_S$-linear homomorphism

$$\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_S) = \theta_S \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{E}), \quad w \mapsto \nabla_w,$$

where ∇_w satisfies the Leibniz identity.

In particular, if $\dim S = 1$ and $w = \mathrm{d}/\mathrm{d}t$ is a basis vector field on S, then giving a connection on $\mathcal{E}$ is equivalent to giving an operator $D = \nabla_{\mathrm{d}/\mathrm{d}t}: \mathcal{E} \to \mathcal{E}$ satisfying the Leibniz identity. The operator D will be also denoted by ∂_t.

(2.2.5) *Description of all connections on* $\mathcal{E}$ If $\nabla_1, \nabla_2 \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{E}, \Omega^1_S(\mathcal{E}))$ are two connections on $\mathcal{E}$, then it follows from the Leibniz identity that their difference $\nabla_2 - \nabla_1 \in \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{E}, \Omega_S(\mathcal{E}))$ is not only a $\mathbb{C}$-linear, but also an $\mathcal{O}_S$-linear homomorphism from $\mathcal{E}$ to $\Omega^1_S(\mathcal{E})$. Conversely, if ∇_0 is a connection on $\mathcal{E}$ and $\Gamma \in \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{E}, \Omega^1_S(\mathcal{E}))$, then $\nabla = \nabla_0 + \Gamma$ is a connection on $\mathscr{E}$. Thus connections on $\mathscr{E}$ form a principal homogeneous space over $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{E}, \Omega^1_S(\mathscr{E}))$.

2.3 Local calculation of connections. Relation to differential equations

(2.3.1) Let $\mathscr{E}$ be a locally free sheaf of rank n. Choose in $\mathscr{E}$ a basis (locally) $e_1, \dots, e_n$, and let $e\colon \mathscr{O}^n_S \simeq \mathscr{E}$, $e\colon (0, \dots, 1, \dots, 0) \mapsto e_i$. For brevity we'll use matrix notation: let $e = (e_1, \dots, e_n)$ be a row of base vectors, and let $y = (y_1(x), \dots, y_n(x))^t$ be a column of coordinates of a section s wrt the basis e. The section s of $\mathscr{E}$ is represented by the scalar product $s = ey$.

A choice of basis e defines a constant subsheaf $e\colon \mathbb{C}^n_S \subset \mathscr{E}$ and a connection ∇_0 on $\mathscr{E}$, $\nabla_0 s = ds$, i.e.

$$\nabla_0 s = e\,\mathrm{d}y = \sum_{i=1}^{n} \mathrm{d}y_i e_i,$$

for which the sections e_i are horizontal, $\nabla_0(e_i) = 0$. Any connection ∇ on $\mathscr{E}$ is represented as

(2.3.2) $$\nabla = \nabla_0 + \Gamma, \quad \text{where} \quad \Gamma \in \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{E}, \Omega^1(\mathscr{E})).$$

In the basis e, the homomorphism $\Gamma\colon \mathscr{E} \to \Omega^1_S \otimes \mathscr{E}$, $\Gamma\colon \mathscr{O}_S e_1 \oplus \cdots \oplus \mathscr{O}_S e_n \to \Omega^1_S e_1 \oplus \cdots \oplus \Omega^1_S e_n$ is given by *a matrix of connection forms* $\omega = (\omega_{ij})$

(2.3.3) $$\Gamma(e_j) = \sum_{i=1}^{n} \omega_{ij} e_i, \quad \text{i.e. } \Gamma(e) = e\omega,$$

i.e. ω is the matrix of the linear transformation Γ in the basis e: the coordinates of vector $\Gamma(e_j)$ are in the jth column of the matrix ω. Now, if $s = ey$, then $\nabla s = \nabla_0 s + \Gamma s$, i.e.

$$e\nabla y = e\,\mathrm{d}y + \Gamma(ey) = e\,\mathrm{d}y + \Gamma(e)y = e\,\mathrm{d}y + e\omega y$$

or in coordinates

$$\nabla y = \mathrm{d}y + \omega y, \quad \nabla \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \mathrm{d}y_1 \\ \vdots \\ \mathrm{d}y_n \end{pmatrix} + (\omega_{ij}) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix},$$

i.e.

(2.3.4) $$(\nabla y)_i = \mathrm{d}y_i + \sum_{j=1}^{n} \omega_{ij} y_j, \quad i = 1, \dots, n.$$

Now let $x_1, \dots, x_m$ be local coordinates on S. Then the connection forms $\omega_{ij} \in \Omega^1_S$ are written in the form

(2.3.5)
$$\omega_{ij} = \sum_{k=1}^{m} \Gamma_{ij}^{k}\, \mathrm{d}x_k,$$

where the holomorphic functions $\Gamma_{ij}^{k}(x)$ are called the *connection coefficients* (wrt the basis e and coordinates x).

Denote by $\partial_i = \partial/\partial x_i$ the differentiation wrt the coordinate x_i, and let $\nabla_i = \nabla_{\partial_i}\colon \mathscr{E} \to \mathscr{E}$ be the covariant derivative wrt the vector field ∂_i. Then (2.3.4) turns into a system of equalities

(2.3.6)
$$(\nabla_k y)_i = \partial y_i/\partial x_k + \sum_{j=1}^{n} \Gamma_{ij}^{k} \partial y_i/\partial x_k, \qquad i = 1, \ldots, n,\ k = 1, \ldots, m$$

(2.3.7) *Definition* A local section s of the sheaf $\mathscr{E}$ is called *horizontal* if $\nabla s = 0$, i.e. $s \in \operatorname{Ker} \nabla$, $\nabla\colon \mathscr{E} \to \Omega_S^1 \otimes \mathscr{E}$.

The condition of horizontality of section s can be written in the form $\nabla_k s = 0$, $k = 0, \ldots, m$, i.e. in the form of system of homogeneous linear differential equations of the first order

(2.3.8)
$$\partial y_i/\partial x_k = -\sum_{j=1}^{n} \Gamma_{ij}^{k} \partial y_j/\partial x_k, \qquad i = 1, \ldots, n,\ k = 1, \ldots, m.$$

This is a system of nm equations with n unknown functions $y_i(x)$ in m variables $x_1, \ldots, x_m$.

If $\dim S = m = 1$, t is a coordinate on S, $\omega_{ij} = \Gamma_{ij}(t)\,\mathrm{d}t$, and $\Gamma = (\Gamma_{ij}(t))$ is the matrix of connection coefficients, then the operator $D = \partial_t$ acts on a section $s = ey$ according to the formula:

(2.3.9)
$$Dy = \mathrm{d}y/\mathrm{d}t + \Gamma(t)y$$

and the condition for horizontality of s is written in the form of a system of ordinary differential equations

(2.3.10)
$$\mathrm{d}y/\mathrm{d}t = -\Gamma(t)y.$$

2.4 The integrable connections. The De Rham complex

(2.4.1) A connection $\nabla = \nabla_0\colon \mathscr{E} \to \Omega_S^1 \otimes \mathscr{E}$ can be extended to a $\mathbb{C}$-linear homomorphism of sheaves

$$\nabla_i\colon \Omega_S^1 \otimes_{\mathscr{O}_S} \mathscr{E} \to \Omega_S^{i+1} \otimes_{\mathscr{O}_S} \mathscr{E}$$

by means of the equalities

$$\nabla_i(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^i \omega \wedge \nabla_0(s).$$

(2.4.2) The $\mathbb{C}$-linear homomorphism

$$R = \nabla_1 \circ \nabla_0 \colon \mathcal{E} \to \Omega^2_S(\mathcal{E})$$

is called the curvature of the connection ∇ on $\mathcal{E}$, so that R is a section of the sheaf $\mathcal{H}om_{\mathbb{C}}(\mathcal{E}, \Omega^2_S(\mathcal{E})) \simeq \Omega^2_S(\mathcal{E}nd_{\mathbb{C}}(\mathcal{E}))$. A connection ∇ is called *integrable* if $R = \nabla_1 \nabla_0 = 0$.

One can show that for vector fields $X_1, X_2 \in \theta_S$ and a section $s \in \mathcal{E}$ one has

$$R(X_1, X_2)(s) = \nabla_{X_1}\nabla_{X_2}s - \nabla_{X_2}\nabla_{X_1}s - \nabla_{[X_1,X_2]}s,$$

and the condition of integrability of connection ∇ is equivalent to the condition $\nabla_{[X,Y]} = [\nabla_X, \nabla_Y]$ for arbitrary vector fields X and Y. If $\dim S = 1$, then any connection is integrable. A local calculation of curvature R shows that the condition of integrability of ∇ is equivalent to the classical condition of integrability of the corresponding system of linear differential equations.

(2.4.3) One can show that $\nabla_{i+1}\nabla_i(\omega \otimes s) = \omega \wedge R(s)$ (the Ricci identity). Therefore the condition of integrability of a connection ∇, $R = 0$, is equivalent to the condition that

$$\Omega^{\cdot}_S(\mathcal{E})\colon 0 \to \mathcal{O}_S \underset{\mathcal{O}_S}{\otimes} \mathcal{E} \xrightarrow{\nabla_0} \Omega^1_S \underset{\mathcal{O}_S}{\otimes} \mathcal{E} \xrightarrow{\nabla_1} \ldots \xrightarrow{\nabla_{m-1}} \Omega^m_S \underset{\mathcal{O}_S}{\otimes} \mathcal{E} \to 0 \tag{2.4.4}$$

is a complex, i.e. $\nabla_{i+1}\nabla_i = 0$. The complex $\Omega^{\cdot}_S(\mathcal{E})$ is called the *De Rham complex* with values in a locally free sheaf $\mathcal{E}$ with an integrable connection ∇.

2.5 Local systems and integrable connections

(2.5.1) Let S be a connected and locally connected topological space. A locally free sheaf of vector spaces E on S, i.e. a sheaf locally isomorphic to a constant sheaf $\mathbb{C}^n$, is called a *local system* on S.

(2.5.2) The fundamental group $\pi_1(S, x_0)$, $x_0 \in S$, acts on the fibre $E_{x_0} \simeq \mathbb{C}^n$. The functor $E \mapsto E_{x_0}$ establishes an equivalence of the category of local systems on S with the category of complex-valued representations of the group $\pi_1(S, x_0)$.

(2.5.3) Now let E be a local system on a complex manifold S. Consider a locally free sheaf $\mathcal{E}$ of holomorphic sections of E, $\mathcal{E} = \mathcal{O}_S \otimes_{\mathbb{C}} E$. Then there is a *canonical connection* ∇ on $\mathcal{E}$ for which the sheaf of horizontal sections coincides with E, $\operatorname{Ker} \nabla = E$. Here ∇ is defined by the formula

$$\nabla(gs) = dg\ s$$

where g and s are sections of $\mathcal{O}_S$ and E. Clearly, the canonical connection is integrable. Conversely, one has the following:

(2.5.4) *Theorem* [D1] Let ∇ be a connection on a locally free sheaf $\mathcal{E}$. Put $E = \operatorname{Ker} \nabla = \{s \in \mathcal{E} | \nabla s = 0\}$. If ∇ is an integrable connection, then E is a local system on X and $\mathcal{E} = \mathcal{O}_S \otimes_{\mathbb{C}} E$. ■

Therefore, the functors:

(a) a local system $E \mapsto$ the sheaf $\mathcal{E} = \mathcal{O}_S \otimes_{\mathbb{C}} E$ and the canonical connection ∇; and
(b) a locally free sheaf $\mathcal{E}$ with an integrable connection $\mapsto E = \operatorname{Ker} \nabla$,

establish an equivalence of categories of local systems on E and locally free sheaves with integrable connections.

2.6 Dual local systems and connections

We intend to study the local system $\underline{H} = R^p f_* \mathbb{C}_{X'} = \bigcup_{t \in S'} H^p(X_t, \mathbb{C})$ on a punctured disk, introduced in §1. $\underline{H}$ is called the *cohomological fibration*. Geometrically the dual *homological fibration* $\underline{H}' = \bigcup H_p(X_t, \mathbb{C})$ appears in a natural way. So we consider the following general situation:

(2.6.1) Let $\underline{H}$ be a local system on, say, the punctured disk S'. Denote by $\underline{H}^* = \operatorname{Hom}_{\mathbb{C}_{S'}}(\underline{H}, \mathbb{C}_{S'})$ the dual local system, let

$$\mathcal{H} = \underline{H} \underset{\mathbb{C}_{S'}}{\otimes} \mathcal{O}_{S'} \quad \text{and} \quad \mathcal{H}^* = \underline{H}^* \underset{\mathbb{C}_{S'}}{\otimes} \mathcal{O}_{S'} = \mathcal{H}om_{\mathcal{O}_{S'}}(\mathcal{H}, \mathcal{O}_{S'})$$

be the locally free sheaves corresponding to $\underline{H}$ and $\underline{H}^*$, and let ∇ and ∇^* be the connections on $\mathcal{H}$ and $\mathcal{H}^*$ defined by the above local systems. The connection ∇^* is called the dual of ∇.

Denote by

$$\langle\ ,\ \rangle : \underline{H} \times \underline{H}^* \to \mathbb{C}_{S'}$$

the associated non-degenerate pairing which extends to a pairing $\mathcal{H} \times \mathcal{H}^* \to \mathcal{O}_{S'}$. Denote by T and M the monodromy transformations on $\underline{H}$ and $\underline{H}^*$, respectively. Obviously, these are related by the formula

$$T = (M^*)^{-1},$$

where M^* denotes linear transformation on the dual space.

(2.6.2) Let us see how the operators of covariant differentiation

$$D = \nabla_{d/dt}: \mathscr{H} \to \mathscr{H} \quad \text{and} \quad D^* = \nabla^*_{d/dt}: \mathscr{H}^* \to \mathscr{H}^*$$

are related to each other.

Proposition 1 If ω is a local section of the sheaf $\mathscr{H}$ and γ is a local section of $\underline{H}^*$, i.e. a horizontal section of $\mathscr{H}^*$, then the derivative of the function $I(t) = \langle \omega, \gamma \rangle$ is

(2.6.3) $$\langle \omega, \gamma \rangle' = \langle D\omega, \gamma \rangle.$$

Proof If the formula is true for ω_1 and ω_2, then it is true for $\omega = \omega_1 + \omega_2$ by virtue of the additivity of both sides of the formula wrt ω. Let us show that if (2.6.3) is true for ω, then it is true for $g\omega$, where $g \in \mathscr{O}_{S'}$:

$$\text{left-hand side } = \langle g\omega, \gamma \rangle' = (g\langle \omega, \gamma \rangle)' = g'\langle \omega, \gamma \rangle + g\langle \omega, \gamma \rangle',$$
$$\text{right-hand side } = \langle Dg\omega, \gamma \rangle = \langle g'\omega + gD\omega, \gamma \rangle = g'\langle \omega, \gamma \rangle + g\langle D\omega, \gamma \rangle.$$

The formula is true for horizontal sections ω, because then $\langle \omega, \gamma \rangle = \text{const}$ and $\langle \omega, \gamma \rangle' = 0$, and on the other hand $D\omega = 0$ and $\langle D\omega, \gamma \rangle = 0$. The formula is true for any section ω because horizontal sections generate $\mathscr{H}$ over $\mathscr{O}_{S'}$. ∎

Proposition 2 If ω and γ are any sections of $\mathscr{H}$ and $\mathscr{H}^*$, then

(2.6.4) $$\langle \omega, \gamma \rangle' = \langle D\omega, \gamma \rangle + \langle \omega, D^*\gamma \rangle.$$

Proof Arguing as above (the formula is true for horizontal sections; it is additive in γ; being true for γ, it is also true for $g\gamma$, $g \in \mathscr{O}_{S'}$), we obtain the proof. ∎

(2.6.5) Now let $\omega_1, \dots, \omega_\mu$ be a local basis of sections of $\mathscr{H}$ and let $\gamma_1 = \omega_1^*, \dots, \gamma_\mu = \omega_\mu^*$ be the dual basis of the sheaf $\mathscr{H}^*$, $\langle \omega_i, \gamma_j \rangle = \delta_{ij}$.

Proposition 3 If $\Gamma = (\Gamma_{ij})$ is the matrix of coefficients of a connection ∇ in a basis $\omega_1, \dots, \omega_\mu$ and $\Gamma^* = (\Gamma^*_{ij})$ is a matrix of coefficients of ∇^* in the dual basis $\gamma_1, \dots, \gamma_\mu$, then

$$\Gamma^* = -\Gamma^t.$$

Proof Recall that Γ and Γ^* are matrices of operators D and D^* in the bases ω and γ:

$$D\omega_j = \sum_{i=1}^{\mu} \Gamma_{ij}\omega_i, \quad D^*\gamma_i = \sum_{j=1}^{\mu} \Gamma^*_{ji}\gamma_j.$$

We have $\langle \omega_j, \gamma_i \rangle' = 0 = \langle D\omega_j, \gamma_i \rangle + \langle \omega_j, D^*\gamma_i \rangle = \Gamma_{ij} + \Gamma^*_{ji}$. ∎

(2.6.6) Let $\omega_1, \ldots, \omega_\mu$ be a basis of local sections of the sheaf $\mathscr{H}$, and let γ be a section of $\mathscr{H}^*$. Then the functions

$$I_1(t) = \langle \omega_1, \gamma \rangle, \ldots, I_\mu(t) = \langle \omega_\mu, \gamma \rangle$$

can be considered as coordinates of γ in the basis dual to $\{\omega_i\}$:

$$\gamma = I_1(t)\gamma_1 + \ldots + I_\mu(t)\gamma_\mu.$$

In particular, if γ is a section of $\underline{H}^*$, i.e. a horizontal section of $\mathscr{H}^*$, then $\nabla^*\gamma = 0$ and the coordinates of γ satisfy the system (2.3.10) of differential equations of the connection ∇^*. Thus if γ is a section of $\underline{H}^*$, then the functions $I_1(t), \ldots, I_\mu(t)$ satisfy a system of differential equations

(2.6.7)
$$\begin{pmatrix} I_1' \\ \vdots \\ I_\mu' \end{pmatrix} = \Gamma^t(t) \begin{pmatrix} I_1 \\ \vdots \\ I_\mu \end{pmatrix},$$

where $\Gamma = (\Gamma_{ij})$ is the matrix of coefficients of ∇ in basis $\omega_1, \ldots, \omega_\mu$.

If $\gamma_1, \ldots, \gamma_\mu$ form a basis of sections of $\underline{H}^*$, then the columns $\Omega_j = (\Omega_{1j}(t), \ldots, \Omega_{\mu j}(t))^t$, $\Omega_{ij} = \langle \omega_i, \gamma_j \rangle$, form a basis of solutions, and the matrix

(2.6.8)
$$\Omega(t) = (\Omega_{ij}(t))$$

is the fundamental matrix of solutions of the system $I' = \Gamma^t(t)I$.

3 De Rham cohomology

3.1 The Poincaré lemma

We return now to our main subject of investigation – the Milnor fibration $f\colon X \to S$ of a singularity of holomorphic function f and the connection on the locally free sheaf $\mathscr{H} = R^p f_* \mathbb{C}_{X'} \otimes \mathscr{O}_{S'}$ on S', defined (topologically) by the local system $R^p f_* \mathbb{C}_{X'}$. In order to pass to differential forms we have to use the Poincaré lemma.

(3.1.1) The Poincaré lemma (in its holomorphic version) asserts that if X is a complex manifold, then the De Rham complex $(\Omega^\cdot_X, \mathrm{d})$ is a resolution of the constant sheaf $\mathbb{C}_X$, i.e. the sequence $0 \to \mathbb{C}_X \to \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \ldots$ is exact.

This implies that for a complex manifold X we have an isomorphism $H^p(X, \mathbb{C}) \simeq \mathbb{H}^p(\Omega^\cdot_X)$, where $\mathbb{H}^p(\Omega^\cdot_X)$ is the hypercohomology of the De Rham complex. If, moreover, X is a Stein manifold, then (see below for the relative case) the cohomology $H^p(X, \mathbb{C})$ is isomorphic to the De Rham