

## Index

- Airy function, 59, 60, 64, 75, 333  
 Airy kernel, 63, 65  
 analytic conductor, 364  
 analytic continuation, 382  
     of  $L$ -functions, 226, 363  
     of the Riemann zeta function, 11  
 analytic properties  
     of  $L$ -functions, 361  
 Andréief's identity, 123, 124, 330  
 approximate functional equation, 190–192, 211, 219, 448, 460  
     for  $\zeta^2(s)$ , 191  
     for  $\zeta^k(s)$ , 191, 218  
     smoothed, 443, 444  
 arithmetic group, 172  
 arithmetic progressions, 6  
 arithmetic-geometric mean inequality, 203, 206, 208  
 Artin conductor, 382  
 Artin conjecture, 382, 383, 385  
 Artin  $L$ -function, 360, 361, 373, 381, 383, 385, 406, 410, 412  
     computation, 470  
     Euler product, 382  
 automorphic forms, 372  
      $L$ -function, 368, 373, 385, 410  
  
 Baker-Campbell-Hausdorff formula, 315  
 Balmer formula, 165, 169  
 Banach algebra, 318, 326  
 Barnes'  $G$ -function, 262, 321, 327  
 Bell number, 339  
 Bernoulli number, 428  
 Bernoulli polynomial, 427  
     special values, 429  
 Bernoulli's formula, 430  
 Bessel function, 333  
 Bessel kernel, 75  
 billiard  
     Bunimovich's stadium, 165–167  
     circle, 165  
     quantum, 166  
     Sinai, 166, 167  
 Birch and Swinnerton-Dyer conjecture, 271, 367, 397, 401, 496  
 Bohigas-Giannoni-Schmit Conjecture, 168  
 Bohr-Sommerfeld quantization, 169  
 Brownian motion, 39, 40  
 Bunimovich's stadium billiard, 165–167  
  
 Cauchy transform of orthogonal polynomials, 72, 74  
 Cauchy's Theorem, 425, 446  
 Cauchy-Schwarz inequality, 209  
 central limit theorem, 256, 261, 265, 271, 334  
 Chadwick's neutron chamber, 151  
 chaotic motion, 164  
 character  
     principal, 230  
 characteristic polynomial, 31, 32, 65–67, 69, 71, 218, 259, 263, 271, 272, 439, 468, 477, 478, 480, 484, 488  
     moments, 195  
 Chebotareff density theorem, 379  
 Chebyshev function, 81  
 Chebyshev polynomial, 119, 121  
 chiral ensemble, 71  
 chiral GUE, 75  
 Christoffel-Darboux, 295  
 Christoffel-Darboux formula, 52, 54, 66  
 circle method, 220  
 circular billiard, 165  
 circular ensembles  
     eigenvalue probability density function, 290  
 Circular Unitary Ensemble, 156, 252, 309  
 class function, 115, 252, 260  
 class number formula, 367  
 classical compact groups, 252, 263, 337, 342, 468, 471, 478, 480, 488  
     Haar measure, 342  
 Classical Mean Value Theorem, 210, 211  
 cluster function, 44, 50, 62, 74  
 coefficient correlation sum, 217  
 compactified  
     elliptic curve, 388  
 complex multiplication, 390  
 compound nucleus hypothesis, 151  
 computing zeros of an  $L$ -function, 466  
 conductor, 364  
     analytic, 364  
     Artin, 382  
     log-conductor, 227, 240  
     of an  $L$ -function, 475  
     of cusp form  $L$ -functions, 475  
     of Dirichlet characters, 228, 453, 470, 474, 478  
     of elliptic curve, 234, 400, 453, 470, 475, 478, 489  
 convexity bound, 367

- convexity estimate, 194  
convolution, 460, 465  
convolution  $L$ -function, 239  
correlation function, 337  
   $n$ -point, 42, 44, 45, 48, 50, 73, 156, 287  
  of characteristic polynomials, 69, 70  
correlations  
  of the zeros of  $L$ -functions, 474  
counting function  
  moments of smooth counting function, 338  
  of eigenvalues, 337  
  of the Riemann zeros,  $N(T)$ , 13, 14, 82, 228, 348, 466  
  of zeros of  $L$ -functions, 337, 467  
Cramér explicit formula, 85  
Cramér's model, 4, 281  
critical line, 11, 28, 66, 104, 107, 158, 160, 173, 185, 186, 189, 191, 201, 204, 206, 207, 209, 261, 263, 268, 299, 415, 442, 466, 478, 488, 491  
critical point, 268, 478, 480, 484, 489  
critical strip, 11, 17, 28, 82, 104, 195, 205, 207, 228, 365–367, 466, 471  
cross section, 151, 152  
CUE, 156, 252  
cumulants, 344  
cusp form, 232, 234, 235, 237, 267, 371, 462  
  primitive, 371  
cusp form  $L$ -function, 352, 447, 474, 475, 478, 479  
  computation, 450  
cyclotomic character, 404  
cyclotomic fields, 378  
Dedekind zeta function, 160, 384, 467  
  computation, 470  
degree of an  $L$ -function, 227, 362, 475  
Deligne's equidistribution theorem, 422  
densities  
   $n$ -level, 337  
density conjecture for low-lying zeros of  $L$ -functions, 240  
density of zeros, 478  
  of an  $L$ -function, 227  
determinant  
  Toeplitz, 309  
difference equation  
  for Bernoulli polynomials, 428  
Dirichlet character, 24, 229, 481  
  orthogonality relations, 230  
  real quadratic, 230, 267  
Dirichlet  $L$ -function, 24, 27, 228, 229, 266, 434, 467, 474–476, 478, 479  
  computation, 450, 470  
  functional equation, 229  
  real characters, 351, 480, 488, 489  
Dirichlet polynomial, 185, 187, 205, 219  
  mean value, 201, 202, 220  
  moments, 187  
Dirichlet series, 202, 361, 443  
  elliptic curve  $L$ -function, 452  
  for  $\zeta(s)^{-1}$ , 82, 204  
  for  $L$ -functions, 226  
  for  $L$ -functions associated to Hecke forms, 451  
  for the Riemann zeta function, 80  
  for  $\zeta^k(s)$ , 218  
  twists of cusp form  $L$ -functions, 451  
  twists of elliptic curve  $L$ -functions, 453  
discrete mean value, 202, 205  
discriminant, 387  
disordered metallic particle, 157, 173  
distribution function  
   $k$ -point, 280, 285, 287  
divisor function, 191, 218  
divisor sum, 220  
dual isogeny, 390  
Dyson kernel, 60, 62, 66, 73  
eigenvalues  
  bulk, 283, 286  
  density for GOE matrices, 283  
  in intervals, 139  
  joint probability density function, 41, 42  
   $j$ th lowest, 142  
  nearest neighbour spacing, 298  
  probability density function, 287, 288  
  circular ensembles, 290  
  symplectic-invariant ensemble, 290  
  statistical properties, 282  
Eisenstein series, 232, 237  
elliptic curve, 267, 273, 386, 387, 389–392, 400, 413, 420, 452, 489  
  compactified, 388  
  complex multiplication, 400  
  conductor, 234, 400, 453, 470, 475, 478, 489  
  family, 248  
  function field, 389, 415  
  Galois representation, 401  
  group law, 388  
   $L$ -function, 234, 248, 267, 360, 386, 398, 470, 475, 476, 478, 479, 498  
  computation, 452, 470  
  vanishing at the central point, 271  
  over finite fields, 393, 396  
  over the rationals, 396

- rank, 248, 396, 397, 401  
reduction, 399  
Tate module, 404
- Essential Simplicity Conjecture, 94
- Estermann phenomenon, 198
- Euler product, 362, 444, 493, 494  
for Artin  $L$ -functions, 382  
for elliptic curve  $L$ -functions, 452  
for  $L$ -functions, 226  
for the Riemann zeta function, 7, 80, 228  
local factor, 362
- Euler totient function,  $\varphi(q)$ , 229
- Euler totient function,  $\varphi(q)$ , 25, 195
- Euler's identity, 432
- Euler-Maclaurin summation, 425, 427, 430, 432, 437
- exceptional Lie groups, 272
- explicit formula, 23, 215, 241  
Cramér, 85  
Montgomery, 87  
Riemann-von Mangoldt, 83  
Weil, 366
- exterior square  $L$ -function, 239
- family of  $L$ -functions, 225, 252, 263, 267, 268, 480, 484, 487, 488
- fast Fourier transform, 425, 443, 468
- Feynman path integral, 169
- finite fields  
elliptic curves, 393, 396
- Fisher-Hartwig conjecture, 326–328, 332, 334
- Fisher-Hartwig symbol, 321, 324, 326, 331, 333
- form factor, 89, 93, 171, 173
- Fourier series  
for the Bernoulli polynomials, 428
- Fourier transform, 438, 465
- Fredholm determinant, 50, 287, 295–299, 309, 310
- Fredholm integral equation, 288
- frobenius, 361, 377–380, 382, 393, 395, 404, 406, 408, 417, 418, 420, 423
- function field, 413  
elliptic curve, 389, 415  
zeta function, 478
- functional equation, 363, 382, 488, 489  
Bernoulli polynomials, 428  
Dirichlet  $L$ -function, 229  
elliptic curve  $L$ -function, 453  
 $L$ -function, 226, 443  
 $L$ -functions associated to Hecke forms, 233, 451  
 $L$ -functions of elliptic curves over finite fields, 398
- Riemann zeta function, 11, 208, 227  
theta-series, 369  
twists of cusp form  $L$ -functions, 452  
twists of elliptic curve  $L$ -functions, 268, 453
- fundamental discriminant, 230, 480, 482, 483, 485, 488, 494
- Fundamental Theorem of Arithmetic, 1
- Galois group, 360, 373, 377–379, 381, 401, 403, 413  
absolute, 379, 401, 405  
arithmetic, 417  
geometric, 417
- Galois representation, 360, 361, 376, 409, 410, 420  
elliptic curve, 401
- gamma distribution,  $\text{Gamma}[s, \sigma]$ , 283
- gap probability, 284, 289, 291, 296, 299  
bulk, 296, 297  
hard edge, 296
- gaps  
between consecutive primes, 7, 85, 98  
between zeros of the Riemann zeta function, 90, 91, 471  
small gaps between zeros, 94
- Gaudin's Lemma, 46, 114, 125–127, 129–133, 140
- Gauss sum, 28, 229
- Gaussian ensemble, 31, 39, 73  
joint distribution of eigenvalues, 153
- Gaussian moments of  $S(T)$ , 104
- Gaussian Orthogonal Ensemble (GOE), 153, 283
- Gaussian Symplectic Ensemble (GSE), 153
- Gaussian Unitary Ensemble (GUE), 31, 54, 75, 153, 332
- Gaussian white noise, 39, 41
- Generalized Lindelöf Hypothesis, 205, 209, 367
- Generalized Prime Number Theorem, 366
- Generalized Riemann Hypothesis, 217, 366
- generating function, 285  
for the Bernoulli polynomials, 428
- geometric frobenius, 417
- Geronimo-Case-Borodin-Okounkov identity, 319
- GOE, 153, 283
- Goldbach's Conjecture, 6
- Gram's identity, 115, 137, 140
- group law  
elliptic curve, 388
- GSE, 153
- GUE, 31, 60, 75, 153

- Haar measure, 34, 115, 119, 124  
 on classical compact groups, 342, 380  
 $SO(2N)$ , 120  
 $SO(2N + 1)$ , 122  
 $U(N)$ , 118, 252, 329, 439, 473, 477  
 $USp(2N)$ , 122
- Hadamard product, 12, 365
- Hamilton equations, 293
- Hankel matrix, 318, 334
- Hankel operator, 312, 317, 318, 323
- Hannay-Ozorio sum-rule, 172
- Hardy space, 310
- Hardy-Littlewood conjecture, 93, 174, 435, 473
- Hasse's bound, 393
- Hasse-Weil conjecture, 453
- Hecke congruence group, 234, 370
- Hecke eigenform, 232, 451
- Hecke  $L$ -functions, 370
- Hecke operator, 371, 372, 451
- Heisenberg matrix mechanics, 169
- helium atom, 169, 173
- Hermite polynomial, 31, 32, 51–54, 58–60, 65, 66, 73  
 integral representations, 53  
 orthogonality relations, 51  
 Plancherel-Rotach asymptotics, 31, 53, 54, 57, 63  
 recurrence relations, 51
- Hermitian matrix, 34, 35
- Hilbert space, 310, 311, 314
- Hilbert-Polya conjecture, 174, 175
- Hilbert-Schmidt norm, 314
- Hilbert-Schmidt operator, 314, 318, 324
- hole probability, 44–46
- holomorphic cusp forms, 352
- hydrogen atom, 169, 173
- ideal, 324, 374  
 prime, 375
- incomplete gamma function, 434, 446, 449, 462  
 computation, 455
- integrable systems, 291
- integral operator, 141, 288  
 kernel, 288
- invariant tori, 164
- isogeny, 389  
 dual, 390
- Jensen's Formula, 202
- joint probability density function, 41, 42, 48, 50, 66, 72, 284
- $j$ th lowest eigenvalue, 142
- $K$ -Bessel function, 462
- KAM Theorem, 164
- Katz-Sarnak philosophy, 114, 252, 263, 268, 269, 338, 478
- Keating-Snaith conjecture, 262, 435
- kernel, 49, 50, 60  
 GUE, 61  
 asymptotic form, 62
- kernel function, 113
- Kloosterman sum, 205, 234, 248
- Kronecker's symbol, 351, 480, 482, 483
- $L$ -function, 161, 175, 226, 268, 468  
 analytic continuation, 226, 363  
 analytic properties, 361  
 Artin, 360, 361, 373, 381, 383, 385, 406, 410, 412  
 computation, 442, 447, 470  
 computing zeros, 466  
 conductor, 475  
 convolution, 239  
 cusp form, 352, 447, 450, 474, 475, 478, 479  
 degree, 227, 362, 475  
 density of zeros, 227  
 Dirichlet, 268, 434, 450, 458, 467, 474–476, 478, 479  
 Dirichlet series, 226  
 elliptic curve, 234, 248, 267, 268, 360, 452, 470, 475, 476, 478, 479, 489, 498  
 Euler product, 226  
 exterior square, 239  
 families, 225, 263, 267, 268, 480, 484, 487, 488  
 $G_2$  symmetry, 274  
 functional equation, 226, 443  
 log-conductor, 227  
 Maass form, 238, 462  
 meromorphic continuation, 443  
 modular form, 368, 458  
 of elliptic curves, 386, 398  
 of elliptic curves over finite fields, 398  
 functional equation, 398  
 of elliptic curves over the rationals, 399  
 primitive, 227, 475  
 real Dirichlet  $L$ -function, 351  
 Selberg class, 225, 229  
 symmetric square, 239  
 twists of cusp forms, 451  
 twists of elliptic curve  $L$ -functions, 267, 453, 488  
 vanishing at the central point, 271  
 zero statistics, 471

- Laguerre ensemble, 333
- Lamb shift, 161
- $\Lambda(n)$ , 8
- Landau formula, 84
- Langlands Philosophy, 10
- Laplace-Beltrami operator, 172
- lattice, 391
- Legendre duplication formula, 444
- Legendre symbol, 267
- level density, 170
- Levinson's method, 206, 207
- Levinson-Montgomery Theorem, 207
- Lie group, 115
- Lindelöf Hypothesis  
Generalised, 367
- linear statistic, 329–333, 343
- Littlewood's Lemma, 203, 205, 208
- log-conductor, 240  
of an  $L$ -function, 227
- logarithmic derivative  
of the Riemann zeta function, 80
- Maass form, 237, 238, 243, 372, 462, 470
- MacDonald constant term identities, 274
- Maslov index, 170
- matrix  
orthogonal, 116  
symplectic, 116  
unitary, 115
- mean density, 43
- mean eigenvalue density, 60
- mean value, 204, 214, 217  
Classical Mean Value Theorem, 210, 211  
Dirichlet polynomial, 201, 202, 210, 220  
discrete, 202, 205  
mollified, 205  
Montgomery-Vaughan Mean Value Theorem, 210, 211, 216, 219
- mean value theorem, 201, 202, 221
- mean value theorem for Dirichlet polynomials, 188
- mean value theorem of Montgomery and Vaughan, 88
- Mellin inversion, 449, 461
- Mellin transform, 23, 263, 440, 460
- meromorphic continuation, 363  
of an  $L$ -function, 443  
of the Riemann zeta function, 431
- Mersenne primes, 6
- microwave cavity, 168
- Möbius function, 82, 204, 229, 437
- Möbius inversion, 425, 435, 437
- mock-Gaussian, 338
- modular form, 10, 232, 370
- modular group, 232
- mollified mean values, 205
- mollify, 205, 208
- moment generating function, 261  
for the logarithm of characteristic polynomials, 259
- moments, 491, 498  
for primes in short intervals, 100  
of characteristic polynomials, 66, 69–71, 195, 439  
 $O(N)$ , 264  
 $U(N)$ , 261  
 $USp(2N)$ , 264  
of Dirichlet  $L$ -functions, 494  
of Dirichlet polynomials, 187  
of  $L$ -functions, 251, 269  
of smooth counting function, 338  
of the logarithm of characteristic polynomials  
 $U(N)$ , 261  
of the logarithm of the Riemann zeta function, 257  
of the Riemann zeta function, 185, 201, 217, 435, 488, 489, 491  
of traces of matrices, 346
- monodromy group, 418
- Montgomery's conjecture, 92, 256, 473
- Montgomery's explicit formula, 87
- Montgomery's Theorem, 87, 89
- Montgomery-Odlyzko law, 161, 174, 299
- Montgomery-Vaughan Mean Value Theorem, 210, 211, 216, 219
- moveable singularities, 292
- $n$ -correlation, 134, 136  
for  $U(N)$ , 132  
 $USp(2N)$ , 133
- $n$ -level densities, 337, 480  
large matrix limit, 130  
 $SO(2N)$ , 131  
 $SO(2N+1)$ , 131  
 $U(N)$ , 129  
 $USp(2N)$ , 131
- $n$ -point correlation function, 42, 44, 45, 48, 50, 73, 156
- nearest neighbour spacing distribution, 152, 166, 167, 477  
for  $U(N)$ , 134  
of eigenvalues, 298, 477  
of Riemann zeros, 477  
of zeros of  $L$ -functions, 478
- newforms, 352
- Nielsen's expansion, 458
- $N(T)$ , 13, 14, 82, 228, 348, 466

- Nuclear Data Ensemble, 163  
nuclear resonances, 152, 161, 163, 166  
number variance, 44, 46, 62
- Odlyzko-Schönhage algorithm, 475  
 $O(N)$ , 263  
one-level density, 225, 247, 343, 485, 487  
one-point correlation function, 71  
operator norm, 325  
orthogonal (even) symmetry, 489  
orthogonal group, 342  
orthogonal matrix, 116  
orthogonal polynomial, 31, 32, 46, 49–51, 65, 66, 69, 70, 491  
  monic, 287  
orthogonal projection, 314  
orthogonal symmetry, 225, 268, 352  
  even, 116  
  odd, 116  
orthogonality relations  
  of Dirichlet characters, 230
- Painlevé III' equation, 297, 302, 303  
Painlevé VI equation, 303  
Painlevé V equation, 293, 302, 303  
Painlevé differential equations, 292  
Painlevé equation, 156, 293  
Painlevé theory, 300  
Painlevé transcendent, 291, 297  
pair correlation, 159, 175  
  for eigenvalues of  $U(N)$ , 131  
  of Riemann zeros, 96, 132, 201, 472, 473, 480  
  of zeros of  $L$ -functions, 474–476  
Pair Correlation Conjecture, 94  
Pauli principle, 162  
periodic orbit theory, 169–171, 173–175  
Perron's formula, 193  
Petersson formula, 234  
Petersson inner product, 451  
Pfaffian, 289  
 $\varphi(q)$ , 229  
 $\varphi(q)$ , 25, 195  
Phragmén-Lindelöf Theorem, 444  
point process, 280, 281, 284  
Poisson distribution, 281  
Poisson process, 281, 286  
Poisson summation, 425, 455  
  numerical integration, 438  
Poisson Summation Formula, 9  
Polya-Vinogradov inequality, 230  
prime, 1  
  average spacing, 7  
  gaps between consecutive primes, 7, 85, 98  
  ideal, 375  
  Mersenne, 6  
  ramified, 231, 377  
  spacing, 280  
  split, 231  
prime ideal, 375  
Prime Number Theorem, 2, 81, 82, 281, 351  
  Generalised, 366  
  proof, 19  
prime twins, 5  
primes as periodic orbits, 174  
primes in short intervals  
  second moment, 100  
primitive character, 453  
primitive cusp forms, 371  
primitive Dirichlet character, 229  
primitive  $L$ -function, 227  
primitive root, 229  
principal character, 25, 230  
probability density function, 32, 38  
  of values of the characteristic polynomial, 440  
  on Hermitian matrices, 36  
  Orstein-Uhlenbeck, 40  
  unitary-invariant, 37, 38  
projective plane, 387
- quadratic Dirichlet  $L$ -function, 489  
quantum chaos, 147, 173  
quantum electrodynamics, 149
- Ramanujan bound, 226  
Ramanujan's tau-function,  $\tau(n)$   
   $L$ -function, 476  
  quadratic twist  $L$ -function, 487  
Ramanujan's tau-function,  $\tau(n)$ , 231  
  computation, 470  
   $L$ -function, 470, 475, 478  
  quadratic twist  $L$ -function, 487  
Ramanujan/Petersson bound, 364  
ramified, 377  
random matrix  
  circular ensembles  
  eigenvalue probability density function, 290  
  complex Hermitian, 31, 153, 286  
  ensemble, 279  
  independent entries, 32, 37, 38  
  invariant ensembles, 32  
  orthogonally-invariant ensemble, 288  
  real quaternion, 153, 286

- real symmetric, 31, 153, 279, 282, 283, 288
- symplectic-invariant ensemble
  - eigenvalue probability density function, 290
- unitary-invariant ensemble, 287
- Wishart, 75
- Random Matrix Conjecture, 168
- random polynomial, 283
- random variable, 329, 331, 332
  - distribution function, 329
- rank
  - of an elliptic curve, 248, 396, 397, 401
- Rankin-Selberg method, 463
- ratios
  - of characteristic polynomials, 71–73, 76
- real Dirichlet  $L$ -function, 351
- reduction of an elliptic curve, 399
- resolvent kernel, 300
- resolvent matrix, 71
- resolvent operator, 295
- resonance peaks, 151
- resonances, 151
- Riccati equation, 293
- Riemann Hypothesis, 11, 82, 174, 204, 205, 207, 209, 228, 256, 268, 348, 444, 466, 471, 484
  - Generalised, 366
  - computational verification, 470
  - for  $L$ -functions of elliptic curves over finite fields, 398
- Riemann sum, 438, 440, 465
- Riemann zeta function, 7, 66, 80, 157, 159, 173, 174, 201, 227, 256, 299, 348, 368, 458, 488
  - analytic continuation, 11
  - approximate formula, 186
  - average value on the critical line, 186
  - computation, 450
  - Dirichlet series, 80
  - Euler product, 7, 80
  - functional equation, 11
  - logarithmic derivative, 80
  - moments, 185
  - trivial zeros, 11
  - zero counting function, 13, 14, 348
- Riemann-Hilbert, 63, 74–76, 157
- Riemann-Siegel formula, 433, 434, 442, 442, 460, 468
- Riemann-von Mangoldt formula, 14, 82
- Riemannian metric, 32–34
- ring of integers, 374
- root number, 364
- Rydberg atom, 173
- saddle-point method, 54, 57, 58
- Sato-Tate law, 270
- Sato/Tate conjecture, 408
- Sato/Tate measure, 380
- Schrödinger equation, 161, 169, 171
- Selberg class, 225, 229
- Selberg integral, 260, 264, 439
- Selberg trace formula, 172
- semiclassical limit, 172
- set partition, 339
- shell-model, 161, 162
- sieve of Eratosthenes, 437
- simple zeros, 209, 210
- Simple Zeros Conjecture, 94
- Sinai's billiard, 166, 167
- sine kernel, 288, 295, 299
- singular symbol, 331
- Small Gaps Conjecture, 94
- smooth counting function, 343, 349
- $SO(2N)$ , 116, 225, 265, 268, 342
- $SO(2N + 1)$ , 116, 225, 342
- $SO(N)$ , 116
- spacing distribution, 279, 280, 283, 286, 296
  - empirical determination, 282
  - in the bulk, 301, 303
- spectral fluctuations, 162
- spectral rigidity, 163, 167
- spectrum
  - bulk, 65, 73
  - edge, 63, 65
  - hard edge, 296, 296
  - soft edge, 296
  - unfold, 282
- Speiser's Theorem, 207
- $S(T)$ , 13, 83, 97, 103, 105, 228, 348, 467
  - Gaussian moments, 104
- stationary phase, 54
- steepest descent, 54
- Stieltjes integral, 426
- Stirling number, 339
- Stirling's formula, 60, 62, 446, 468
- Strong Pair Correlation Conjecture, 93
- Strong Szegő Limit Theorem, 319–322, 330–332
- subconvexity, 368
- summation by parts, 426
- supersymmetric non-linear  $\sigma$ -model, 173
- supersymmetry, 491
- symbol, 313
  - Fisher-Hartwig, 321, 324, 326, 331, 333
  - singular, 309, 331
  - smooth, 309
- symmetric square  $L$ -function, 239
- symplectic group, 117

- symplectic matrix, 116  
 symplectic symmetry, 225, 268, 351, 488  
 Tamagawa factor, 496  
 Tate module, 402, 405  
 Tate-Shafarevich group, 496  
 Tchebychev polynomial, 119  
 theta-function,  $\theta(v)$ , 10  
 theta-series, 368  
 Toeplitz determinant, 309, 311, 329–331  
 Toeplitz matrix, 309, 310, 313, 322, 327, 332, 334  
 Toeplitz operator, 311–314, 324  
 torsion subgroup, 396  
 torus quantization, 173  
 trace class, 314–317, 323, 325  
 trace norm, 314, 315, 323, 325  
 transposing lemma, 118, 120, 122  
 trivial zeros  
     of the Riemann zeta function, 11, 228  
 Twin Prime Conjecture, 216  
 twin primes, 5, 101  
 twists of cusp form  $L$ -functions  
     computation, 451  
 twists of elliptic curve  $L$ -functions  
     computation, 453  
 two-body random ensemble, 162  
 two-norm, 311  
 two-point correlation function, 43, 50, 171, 173, 253  
     of eigenvalues from the CUE ( $U(N)$ ), 252  
     of the Riemann zeros, 256  
 $U(N)$ , 115, 156, 225, 252, 342  
 unfold, 283  
 unilateral sequence, 310  
 unitary group, 252, 342  
 unitary symmetry, 225, 348, 488  
 unitary symplectic group, 342  
 unitary matrix, 115  
 $USp(2N)$ , 117, 225, 263, 268, 342  
 value distribution, 489, 490, 492  
     of characteristic polynomials  
          $O(N)$ , 264  
          $SO(2N)$ , 266  
          $U(N)$ , 263, 264  
          $USp(2N)$ , 264, 265  
     of Dirichlet  $L$ -functions, 269  
     of elliptic curve  $L$ -functions, 269, 271  
     of  $L$ -functions, 488  
     of the logarithm of characteristic polynomials  
         of  $U(N)$ , 260, 261  
         of the logarithm of the Riemann zeta function, 256, 257  
         of the Riemann zeta function, 256, 264  
 Vandermonde determinant, 48, 67, 69, 117, 329  
 Virasoro constraints, 300  
 von Mangoldt, 366  
 von Mangoldt function,  $\Lambda(n)$ , 8, 81, 195  
 Weierstrass  $\mathcal{P}$ -function, 293  
 Weierstrass form, 452  
 weight, 364  
 Weil explicit formula, 366  
 Weil pairing, 405  
 Weyl equidistribution criterion, 381  
 Weyl integration formula, 130, 252, 254, 260, 264, 274, 439  
      $SO(2N)$ , 116  
      $SO(2N+1)$ , 116  
      $U(N)$ , 115  
      $USp(2N)$ , 117  
 white noise  
     Gaussian, 39, 41  
 Wiener-Hopf operator, 332  
 Wigner semi-circle law, 60, 61, 156, 162, 283  
 Wigner surmise, 153, 284, 286, 290, 303  
 zero-density estimate, 205, 206  
 zero-free region, 367  
 zeros  
     near the critical point, 478  
     of  $L$ -functions  
         computation, 466  
     of the Riemann zeta function, 207  
     simple, 209, 210  
     spacing between zeros of the Riemann zeta function, 471  
 $\zeta(s)^{-1}$ , 82, 204  
 $\zeta(s)$ , 80, 227