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Recent Perspectives in Random Matrix Theory and Number Theory

Edited by

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Introduction

F. Mezzadri and N. C. Snaith

This volume of proceedings stems from a school that was part of the programme *Random Matrix Approaches in Number Theory*, which ran at the Isaac Newton Institute for Mathematical Sciences, Cambridge, from 26 January until 16 July 2004. The purpose of these proceedings is twofold. Firstly, the impressive recent progress in analytic number theory brought about by the introduction of random matrix techniques has created a rapidly developing area of research. As a consequence there is not as yet a textbook on the subject. This volume is intended to fill this gap. There are, of course, well-established texts in both random matrix theory and analytic number theory, but very few of them treat in any length or detail these new applications of random matrix theory. Secondly, this new branch of mathematics is intrinsically multidisciplinary; teaching young researchers in random matrix theory, mathematical physics and number theory mathematical techniques that are not a natural part of their education is essential to introduce a new generation of scientists to this important and rapidly developing field. In writing their contributions to the proceedings, the lecturers kept in mind the diverse backgrounds of the audience to whom this volume is addressed.

The material in the volume includes the basic techniques of random matrix theory and number theory needed to understand the most important achievements in the subject; it also gives a comprehensive survey of recent results where random matrix theory has played a major role in advancing our understanding of open problems in number theory. We hope that the choice of topics will be useful both to the advanced graduate student and to the established researcher.

These proceedings contain a set of introductory lectures to analytic number theory and random matrix theory, written by Roger Heath-Brown and Yan Fyodorov respectively. The former includes a survey of elementary prime number theory and an introduction to the theory of the Riemann zeta function and other L-functions, while Fyodorov's lectures provide the reader with one of the main tools used in the theory of random matrices: the theory of orthogonal polynomials. This ubiquitous technique is then applied to the computation of the spectral correlation functions of eigenvalues

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of the Hermitian matrices which form the Gaussian Unitary Ensemble (GUE), as well as to computing the averages of moments and ratios of characteristic polynomials of these Hermitian matrices. In contrast, fundamental techniques for calculating various eigenvalue statistics on ensembles of unitary matrices can be found in the "Notes on eigenvalue distributions for the classical compact groups" by Brian Conrey. These are the groups of matrices that are used in connection with L-functions, for example in the lectures of Hughes and Keating. The articles of Peter Forrester and Estelle Basor discuss more specific topics in random matrix theory. Forrester reviews in detail the theory of spacing distributions for various ensembles of matrices and emphasizes its connections with the theory of Painlevé equations and of Fredholm determinants; Basor's lectures introduce the reader to the theory of Toeplitz determinants, their asymptotic evaluations for both smooth and singular symbols and their connection to random matrix theory.

Dan Goldston reviews how random matrix theory and number theory came together unexpectedly when Montgomery, assuming the Riemann hypothesis, conjectured the two-point correlation function of the Riemann zeros, which Dyson recognized as the two-point correlation function for eigenvalues of the random matrices in the CUE (or, equivalently, the GUE) ensemble. Looking toward applications to physics, Oriol Bohigas's article gives an historical survey of how random matrix theory was instrumental in the understanding of the statistical properties of spectra of complex nuclei and of individual quantum mechanical systems whose classical limit exhibits chaotic behaviour. After Montgomory's discovery overwhelming numerical evidence, largely produced by Andrew Odlyzko in the late 1980s, supported the hypothesis that the nontrivial zeros of the Riemann zeta function are locally correlated like eigenvalues of random matrices in the GUE ensemble. Later Hejhal (1994), and then Rudnick and Sarnak (1994,1996) proved similar results for the three and higher point correlations.

Several lectures are devoted to specific and more advanced topics in number theory. David Farmer introduces the reader to techniques in analytic number theory, discussing various ways to manipulate Dirichlet series, while Steve Gonek extends this to discuss mean-value theorems and their applications. Philippe Michel discusses the construction of many examples of L-functions, including those associated to elliptic curves and modular forms.

The remaining lectures highlight the connection between L-functions and random matrix theory. Brian Conrey's lectures "Families of L-functions and 1-level densities" concern the statistics of zeros of families of L-functions near the point where the line on which their Riemann hypothesis places their zeros crosses the real axis. Based on the example of the function field zeta functions, these statistics were proposed by Katz and Sarnak (1999) to be those of the eigenvalues of one of the classical compact groups, namely U(N), USp($2N$) and O(N). The lectures of Jon Keating reveal how the local statistical properties of the Riemann zeta function and other L-functions are inherently determined by the distribution of their zeros, thus high up the critical line

$\zeta(s)$ can be modelled by the characteristic polynomial of random matrices belonging to U(N). As a consequence of this property, techniques well developed in random matrix theory can lead to conjectures for quantities like moments and distributions of the values of L-functions, which have been open problems for almost eighty years. Chris Hughes discusses how the first few moments of the smooth counting functions of the eigenvalues of random matrices and of the zeros of L-functions are Gaussian while their distributions are not. Since much of the predictive power of random matrix theory is based on conjectures, numerical experiments play an important role in the theory; Michael Rubinstein's article introduces the reader to the most important techniques used in computational number theory and to conjectures and numerical experiments connecting number theory with random matrix theory.

We are particularly grateful to David Farmer and Brian Conrey for carefully reading many of the articles and to the staff of the Newton Institute for their invaluable assistance in making the school such a successful event. We also thankfully acknowledge financial contributions from the EU Network 'Mathematical Aspects of Quantum Chaos', the Institute of Physics Publishing, the Isaac Newton Institute for the Mathematical Sciences and the US National Science Foundation.

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