

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)

RIEMANNIAN GEOMETRY

*A Modern Introduction**Second Edition*

This book provides an introduction to Riemannian geometry, the geometry of curved spaces, for use in a graduate course. Requiring only an understanding of differentiable manifolds, the book covers the introductory ideas of Riemannian geometry, followed by a selection of more specialized topics. Also featured are Notes and Exercises for each chapter to develop and enrich the reader's appreciation of the subject. This second edition has a clearer treatment of many topics from the first edition, with new proofs of some theorems. Also a new chapter on the Riemannian geometry of surfaces has been added.

The main themes here are the effect of curvature on the usual notions of classical Euclidean geometry, and the new notions and ideas motivated by curvature itself. Among the classical topics shown in a new setting is isoperimetric inequalities – the interplay of volume of sets and the areas of their boundaries – in curved space. Completely new themes created by curvature include the classical Rauch comparison theorem and its consequences in geometry and topology, and the interaction of microscopic behavior of the geometry with the macroscopic structure of the space.

Isaac Chavel is Professor of Mathematics at The City College of the City University of New York. He received his Ph.D. in Mathematics from Yeshiva University under the direction of Professor Harry E. Rauch. He has published in international journals in the areas of differential geometry and partial differential equations, especially the Laplace and heat operators on Riemannian manifolds. His other books include *Eigenvalues in Riemannian Geometry* (1984) and *Isoperimetric Inequalities: Differential Geometric and Analytic Perspectives* (2001). He has been teaching at The City College of the City University of New York since 1970, and he has been a member of the doctoral program of the City University of New York since 1976. He is a member of the American Mathematical Society.

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board:

B. Bollobás, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

Already published

- 17 W. Dicks & M. Dunwoody *Groups acting on graphs*
- 18 L.J. Corwin & F.P. Greenleaf *Representations of nilpotent Lie groups and their applications*
- 19 R. Fritsch & R. Piccinini *Cellular structures in topology*
- 20 H. Klingen *Introductory lectures on Siegel modular forms*
- 21 P. Koosis *The logarithmic integral II*
- 22 M.J. Collins *Representations and characters of finite groups*
- 24 H. Kunita *Stochastic flows and stochastic differential equations*
- 25 P. Wojtaszczyk *Banach spaces for analysis*
- 26 J.E. Gilbert & M.A.M. Murray *Clifford algebras and Dirac operators in harmonic analysis*
- 27 A. Fröhlich & M.J. Taylor *Algebraic number theory*
- 28 K. Goebel & W.A. Kirk *Topics in metric fixed point theory*
- 29 J.F. Humphreys *Reflection groups and Coxeter groups*
- 30 D.J. Benson *Representations and cohomology I*
- 31 D.J. Benson *Representations and cohomology II*
- 32 C. Allday & V. Puppe *Cohomological methods in transformation groups*
- 33 C. Soulé et al. *Lectures on Arakelov geometry*
- 34 A. Ambrosetti & G. Prodi *A primer of nonlinear analysis*
- 35 J. Palis & F. Takens *Hyperbolicity, stability and chaos at homoclinic bifurcations*
- 37 Y. Meyer *Wavelets and operators I*
- 38 C. Weibel *An introduction to homological algebra*
- 39 W. Bruns & J. Herzog *Cohen–Macaulay rings*
- 40 V. Snaith *Explicit Brauer induction*
- 41 G. Laumon *Cohomology of Drinfeld modular varieties I*
- 42 E.B. Davies *Spectral theory and differential operators*
- 43 J. Diestel, H. Jarchow, & A. Tonge *Absolutely summing operators*
- 44 P. Mattila *Geometry of sets and measures in Euclidean spaces*
- 45 R. Pinsky *Positive harmonic functions and diffusion*
- 46 G. Tenenbaum *Introduction to analytic and probabilistic number theory*
- 47 C. Peskine *An algebraic introduction to complex projective geometry*
- 48 Y. Meyer & R. Coifman *Wavelets*
- 49 R. Stanley *Enumerative combinatorics I*
- 50 I. Porteous *Clifford algebras and the classical groups*
- 51 M. Audin *Spinning tops*
- 52 V. Jurdjevic *Geometric control theory*
- 53 H. Volklein *Groups as Galois groups*
- 54 J. Le Potier *Lectures on vector bundles*
- 55 D. Bump *Automorphic forms and representations*
- 56 G. Laumon *Cohomology of Drinfeld modular varieties II*
- 57 D.M. Clark & B.A. Davey *Natural dualities for the working algebraist*
- 58 J. McCleary *A user's guide to spectral sequences II*
- 59 P. Taylor *Practical foundations of mathematics*
- 60 M.P. Brodmann & R. Y. Sharp *Local cohomology*
- 61 J.D. Dixon et al. *Analytic pro-p groups*
- 62 R. Stanley *Enumerative combinatorics II*
- 63 R.M. Dudley *Uniform central limit theorems*
- 64 J. Jost & X. Li-Jost *Calculus of variations*
- 65 A.J. Berrick & M.E. Keating *An introduction to rings and modules*
- 66 S. Morosawa *Holomorphic dynamics*
- 67 A.J. Berrick & M.E. Keating *Categories and modules with K-theory in view*
- 68 K. Sato *Levy processes and infinitely divisible distributions*
- 69 H. Hida *Modular forms and Galois cohomology*
- 70 R. Iorio & V. Iorio *Fourier analysis and partial differential equations*
- 71 R. Blei *Analysis in integer and fractional dimensions*
- 72 F. Borceaux & G. Janelidze *Galois theories*
- 73 B. Bollobás *Random graphs*
- 74 R.M. Dudley *Real analysis and probability*
- 75 T. Sheil-Small *Complex polynomials*

(continued on overleaf)

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)Series list (*continued*)

- 76 C. Voisin *Hodge theory and complex algebraic geometry, I*
- 77 C. Voisin *Hodge theory and complex algebraic geometry, II*
- 78 V. Paulsen *Completely bounded maps and operator algebras*
- 79 F. Gesztesy & H. Holden *Soliton equations and their algebro-geometric solutions*
- 81 S. Mukai *An Introduction to invariants and moduli*
- 82 G. Tourlakis *Lectures in logic and set theory I*
- 83 G. Tourlakis *Lectures in logic and set theory II*
- 84 R. Bailey *Association schemes*
- 85 J. Carlson, S. Müller-Stach, & C. Peters *Period mappings and period domains*
- 86 J. Duistermaat & J. Kolk *Multidimensional real analysis I*
- 87 J. Duistermaat & J. Kolk *Multidimensional real analysis II*
- 89 M. Golumbic & A. Trenk *Tolerance graphs*
- 90 L. Harper *Global methods for combinatorial isoperimetric problems*
- 91 I. Moerdijk & J. Mrcun *Introduction to foliations and lie groupoids*
- 92 J. Kollar, K. Smith & A. Corti *Rational and nearly rational varieties*
- 93 D. Applebaum *Lévy processes and stochastic calculus*
- 95 M. Schechter *An introduction to nonlinear analysis*

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)

RIEMANNIAN GEOMETRY

A Modern Introduction

Second Edition

ISAAC CHAVEL

Department of Mathematics

The City College of the

City University of New York



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
 0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition
 Isaac Chavel
 Frontmatter
[More information](#)

CAMBRIDGE UNIVERSITY PRESS
 Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
 40 West 20th Street, New York, NY 10011-4211, USA
www.cambridge.org
 Information on this title: www.cambridge.org/9780521853682

© Cambridge University Press 1994, 2006

This publication is in copyright. Subject to statutory exception
 and to the provisions of relevant collective licensing agreements,
 no reproduction of any part may take place without
 the written permission of Cambridge University Press.

First published 1994
 Second edition published 2006

Printed in the United States of America

A catalog record for this publication is available from the British Library.

Library of Congress Cataloging in Publication Data

Chavel, Isaac.
 Riemannian geometry : a modern introduction / Isaac Chavel. – 2nd ed.
 p. cm. – (Cambridge studies in advanced mathematics; 98)
 Includes bibliographical references and indexes.
 ISBN-13: 978-0-251-85368-2 (alk. paper)
 ISBN-10: 0-521-85368-0 (alk. paper)
 ISBN-13: 978-0-521-61954-7 (pbk. : alk. paper)
 ISBN-10: 0-521-61954-8 (pbk. : alk. paper)
 1. Geometry, Riemannian. I. Title. II. Series.
 QA649.C45 2006
 516.3'73 – dc22 2005029338

ISBN-13 978-0-521-85368-2 hardback
 ISBN-10 0-521-85368-0 hardback

ISBN-13 978-0-521-61954-7 paperback
 ISBN-10 0-521-61954-8 paperback

Cambridge University Press has no responsibility for
 the persistence or accuracy of URLs for external or
 third-party Internet Web sites referred to in this publication
 and does not guarantee that any content on such
 Web sites is, or will remain, accurate or appropriate.

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)

for
HARRY ERNEST RAUCH
(1925–1979)

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)

Contents

	<i>Preface to the Second Edition</i>	page xiii
	<i>Preface</i>	xv
I	Riemannian Manifolds	1
	I.1 Connections	3
	I.2 Parallel Translation of Vector Fields	7
	I.3 Geodesics and the Exponential Map	9
	I.4 The Torsion and Curvature Tensors	13
	I.5 Riemannian Metrics	16
	I.6 The Metric Space Structure	19
	I.7 Geodesics and Completeness	26
	I.8 Calculations with Moving Frames	29
	I.9 Notes and Exercises	32
II	Riemannian Curvature	56
	II.1 The Riemann Sectional Curvature	58
	II.2 Riemannian Submanifolds	60
	II.3 Spaces of Constant Sectional Curvature	64
	II.4 First and Second Variations of Arc Length	73
	II.5 Jacobi's Equation and Criteria	77
	II.6 Elementary Comparison Theorems	84
	II.7 Jacobi Fields and the Exponential Map	88
	II.8 Riemann Normal Coordinates	90
	II.9 Notes and Exercises	94
III	Riemannian Volume	111
	III.1 Geodesic Spherical Coordinates	112
	III.2 The Conjugate and Cut Loci	114
	III.3 Riemannian Measure	119
	III.4 Volume Comparison Theorems	127
	III.5 The Area of Spheres	136
	III.6 Fermi Coordinates	138

x	<i>Contents</i>	
	III.7 Integration of Differential Forms	146
	III.8 Notes and Exercises	154
	III.9 Appendix: Eigenvalue Comparison Theorems	171
IV	Riemannian Coverings	188
	IV.1 Riemannian Coverings	189
	IV.2 The Fundamental Group	195
	IV.3 Volume Growth of Riemannian Coverings	199
	IV.4 Discretization of Riemannian Manifolds	207
	IV.5 The Free Homotopy Classes	217
	IV.6 Notes and Exercises	219
V	Surfaces	229
	V.1 Systolic Inequalities	230
	V.2 Gauss–Bonnet Theory of Surfaces	235
	V.3 The Collar Theorem	244
	V.4 The Isoperimetric Problem: Introduction	247
	V.5 Surfaces with Curvature Bounded from Above	251
	V.6 The Isoperimetric Problem on the Paraboloid of Revolution	267
	V.7 Notes and Exercises	274
VI	Isoperimetric Inequalities (Constant Curvature)	280
	VI.1 The Brunn–Minkowski Theorem	281
	VI.2 Solvability of a Neumann Problem in \mathbb{R}^n	284
	VI.3 Fermi Coordinates in Constant Sectional Curvature Spaces	286
	VI.4 Spherical Symmetrization and Isoperimetric Inequalities	288
	VI.5 M. Gromov’s Uniqueness Proof – Euclidean and Hyperbolic Space	294
	VI.6 The Isoperimetric Inequality on Spheres	297
	VI.7 Notes and Exercises	300
VII	The Kinematic Density	307
	VII.1 The Differential Geometry of Analytical Dynamics	308
	VII.2 The Berger–Kazdan Inequalities	319
	VII.3 On Manifolds with No Conjugate Points	329
	VII.4 Santalo’s Formula	338
	VII.5 Notes and Exercises	342
VIII	Isoperimetric Inequalities (Variable Curvature)	350
	VIII.1 Croke’s Isoperimetric Inequality	351
	VIII.2 Buser’s Isoperimetric Inequality	353
	VIII.3 Isoperimetric Constants	361

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)

<i>Contents</i>		xi
	VIII.4 Discretizations and Isoperimetry	374
	VIII.5 Notes and Exercises	382
IX	Comparison and Finiteness Theorems	385
	IX.1 Preliminaries	385
	IX.2 H. E. Rauch's Comparison Theorem	387
	IX.3 Comparison Theorems with Initial Submanifolds	390
	IX.4 Refinements of the Rauch Theorem	396
	IX.5 Triangle Comparison Theorems	399
	IX.6 Convexity	403
	IX.7 Center of Mass	407
	IX.8 Cheeger's Finiteness Theorem	408
	IX.9 Notes and Exercises	419
	<i>Hints and Sketches for Exercises</i>	427
	<i>Hints and Sketches: Chapter I</i>	427
	<i>Hints and Sketches: Chapter II</i>	429
	<i>Hints and Sketches: Chapter III</i>	432
	<i>Hints and Sketches: Chapter IV</i>	440
	<i>Hints and Sketches: Chapter V</i>	442
	<i>Hints and Sketches: Chapter VI</i>	443
	<i>Hints and Sketches: Chapter VII</i>	444
	<i>Hints and Sketches: Chapter VIII</i>	445
	<i>Hints and Sketches: Chapter IX</i>	446
	<i>Bibliography</i>	449
	<i>Author Index</i>	465
	<i>Subject Index</i>	468

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)

Preface to the Second Edition

In this second edition, the first order of business has been to correct mistakes, mathematical and typographical, large and small, and clarify a number of arguments that were unclear or given short shrift the first time round. I can only hope that, in this process, and in the process of changes and additions described below, I have not introduced any new errors.

I have added some proofs of theorems, and sketches to some of the exercises, that were originally left completely to the reader in the first edition. I have added some new notes and exercises as well.

In the text itself, I have made a few changes. I added a chapter with topics from surfaces, immediately following the chapter on coverings (Chapter IV). The chapter (Chapter V) now includes the Gauss–Bonnet theorem; but, it also contains topics of current interest, showing that the Riemannian geometry of surfaces is alive and well, and is a constant testing ground, as well as a source, of new ideas. As it contained the introduction to the isoperimetric problem in Riemannian manifolds, presenting the Bol–Fiala inequalities, and the Benjamini–Cao solution of the isoperimetric problem on the paraboloid of revolution, I thought it best to follow the chapter with isoperimetric inequalities in the classical constant curvature space forms (Chapter VI).

This last chapter (Chapter VI) is a bit different from what I presented in the first edition. New proofs were given for the isoperimetric problem in Euclidean space, with the famous proof by M. Gromov, using Stokes' theorem, now appearing in my other book *Isoperimetric Inequalities: Differential Geometric and Analytic Perspectives* (2001). The Brunn–Minkowski inequalities in hyperbolic space and the sphere were redone, hopefully improving on the first presentation.

Chapter VI is followed by the original (now Chapter VII) on the kinematic density, with little change. I was sorely tempted to include the Burago–Ivanov solution to the E. Hopf conjecture that metrics on the torus, of *all* dimensions, without conjugate points are flat. But, such an undertaking would have taken the

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)

xiv

Preface to the Second Edition

discussion too far afield. This chapter is then followed by the one on isoperimetric inequalities in general Riemannian manifolds, and the chapter on the Rauch comparison theorem and its consequences.

Beyond the Notes and Exercises sections that conclude each chapter, the reader is highly recommended to M. Berger's recent survey *A Panoramic View of Riemannian Geometry* (2003), preceded by his preparatory essay *Riemannian Geometry During the Second Half of the Twentieth Century* (2000). Just about every page of this introduction to Riemannian geometry could have contained references to Berger's surveys for further background and future work.

It is a pleasure to thank the readers of my first edition for their warm reception of the book and for the helpful criticisms – both in pointing out errors and in suggesting improvements. I should add that I found the reviews very helpful, and I am grateful for the effort that went into them. I hope this edition merits the effort they invested.

ISAAC CHAVEL
Riverdale, New York
February 2005

Preface

My goals in this book on Riemannian geometry are essentially the same as those that guided me in my *Eigenvalues in Riemannian Geometry* (1984): to introduce the subject, to coherently present a number of its basic techniques and results with a mind to future work, and to present some of the results that are attractive in their own right. This book differs from *Eigenvalues* in that it starts at a more basic level. Therefore, it must present a broader view of the ideas from which all the various directions emerge. At the same time, other treatments of Riemannian geometry are available at varying levels and interests, so I need not introduce everything. I have, therefore, attempted a viable introduction to Riemannian geometry for a very broad group of students, with emphases and developments in areas not covered by other books.

My treatment presupposes an introductory course on manifolds, the construction of associated tensor bundles, and Stokes' theorem. When necessary, I recall the facts and/or refer to the literature in which these matters are discussed in detail.

I have not hesitated to prove theorems more than once, with different points of view and arguments. Similarly, I often prove weaker versions of a result and then follow with the stronger version (instead of just subsuming the former result under the latter). The variety of levels, ideas, and approaches is a hallmark of mathematics; and an introductory treatment should display this variety as part of the development of broad technique and as part of the aesthetic appreciation of the mathematical endeavor.

I am confident that a short course could be easily crafted from Chapters I to IV and VII (the second edition: Chapter IX), and a more ambitious course from the remaining material. Every chapter of the book features a Notes and Exercises section. These sections cover (i) references to earlier literature and to other results; (ii) "toes in the water" introductions to topics emerging from the ideas presented in the main body of the text; and (iii) examples and applications. The Notes and Exercises sections of the first four chapters are quite extensive.

Cambridge University Press

0521619548 - Riemannian Geometry: A Modern Introduction, Second Edition

Isaac Chavel

Frontmatter

[More information](#)

xvi

Preface

These sections in the later chapters are not as ambitious as those in the first four, since the first four chapters are genuinely introductory.

The Notes and Exercises sections are organized loosely under subheadings of topics. These are not to be taken too literally; rather, they attempt to restrain the variety of material in these sections from becoming chaotic.

A submotif in the Notes and Exercises sections is the method of calculation with moving frames, even though the method is not used extensively in the main body of the text itself. Besides the obvious claim that such calculations should be included in an introductory treatment, I had in mind a quiet tribute to the late William F. Pohl. I learned the magic long ago of *repère mobile* from Bill Pohl at the University of Minnesota. I can still see his full frame at the blackboard, extending his arm gracefully in front of him, moving his hands descriptively with his fingers playing the role of the frame vectors, and declaring that $\omega_2^3 = 0$ since the frame vector field e_2 did not turn in the direction of e_3 .¹

It is a pleasure to thank my colleagues and friends for their contributions to my work, in general, and to this book, in particular. P. Buser provided me with some helpful discussions and read portions of the work. So did J. Dodziuk and E. A. Feldman. Finally, I wish to thank the geometers of the doctoral program of the City University of New York, namely J. Dodziuk, L. Karp, B. Randol, R. Sacksteder, J. Velling, and Edgar A. Feldman – who have provided, over the years, all sorts of help, mathematical stimulation and insight, and scientific partnership. Their contribution permeates all the pages of this book.

ISAAC CHAVEL
Riverdale, New York
July 1992

¹ My memory hits the mark. As soon as I mentioned to Ed Feldman that I put moving frames in the book because of Bill Pohl, Ed performed an imitation of the grand gesture that was Bill's trademark.