I

Riemannian Manifolds

One cannot start discussing Riemannian geometry without mention of the classics. By "the classics," we refer to the essays of C. F. Gauss (1825, 1827) and B. Riemann (1854), to G. Darboux's summary treatise (1894) of the work of the nineteenth century (and beginning of the twentieth), and to E. Cartan's lectures (1946) in which the method of moving frames became a powerful exciting tool of differential geometry.

Nor may one forget to recommend to the reader the delightful discussion of differential geometry in D. Hilbert–S. Cohn-Vossen (1952).

H. Hopf's notes (1946, 1956) remain eminently readable. A very helpful collection of more current introductory essays is the *MAA Studies* volume edited by S. S. Chern (1989).

In addition, one should refer to the "introductory" five-volume opus of M. Spivak (1970) – wherein the practice of differential geometry is presented in loving detail.

Most recently, one has a definitive overview of the subject at the end of the twentieth century by M. Berger (2003).

Our treatment here is mostly inspired by, and follows in many respects, J. Milnor's elegant and exceptionally clear lecture notes Milnor (1963).[1]

A short summary of the progression of ideas of this chapter is as follows.

Whereas one has, given a differentiable manifold, a natural differentiation of functions on the manifold, one does not have a naturally determined method of differentiation of vector fields on the manifold. Therefore, one considers all possibilities of such differentiation – connections on the manifold. Once one actually picks such a differentiation procedure (i.e., a connection), one determines differentiation of vector fields along paths in the manifold. In particular,

[1] See Note 1 in §I.9.

one has an acceleration vector field (the derivative of the velocity vector field) associated with each C^2 path in the manifold. "Straight lines," usually referred to as *geodesics*, are then the paths in the manifold for which the acceleration is zero – they are the collection of paths describing the "law of inertia" for the manifold with the given connection.

The exponential map (the name inspired by analogy to the exponential map in Lie Theory) then provides a map from the tangent space of any given point of the manifold to the manifold itself, in which lines emanating from the origin of the tangent space are mapped to geodesics in the manifold itself emanating from the point in question. It is in this context that we introduce the torsion and curvature tensors of a connection. For the torsion and curvature tensors arise from the linearization of the differential equations of geodesics; therefore, they will ultimately play a role in studying the differential of the exponential map – the precise role to be explicated in detail in later chapters.

Next, we introduce the Riemannian metrics, the ability to calculate the length of paths in the manifold and to calculate angles of tangent vectors in the same tangent space of the manifold. Again, the specification of the Riemannian metric is not uniquely determined. However, once one has such a metric, one automatically has a preferred connection associated with it. It will always be assumed, unless some explicit comment is made to the contrary, that this connection – the *Levi-Civita connection* – is the one under consideration when examining a given Riemannian metric.

The ability to determine lengths of paths in the manifold then induces a natural metric space structure on the manifold. Namely, the distance between any two points of the manifold is the infimum of the length of all paths connecting the two points. One has the classical theorems that (i) if a path between two points has length equal to the distance between them, then the path may be reparameterized to be a geodesic, and (ii) given any point in the manifold, the point has a neighborhood for which there is one and only one distance minimizing geodesic connecting the original point to any other point in the neighborhood. (Actually, more is true – see §I.6.) This development of ideas concludes (§I.7) with the full characterization of the completeness of the metric structure of the Riemannian metric in terms of the infinite extendability of the geodesics of the Riemannian metric.

The chapter closes with a discussion of calculations using moving frames. We do not really present any new material; rather, we revisit some of the previous calculations with a new tool to be developed in its own right and to be used later on.

§I.1. Connections

We refer the reader to Narasimhan (1968) and Warner (1971) for background on differentiable manifolds. Needless to say, these are not the only possible quality choices.

For a path $\omega(t)$ in a manifold M, we let $\omega'(t)$ denote the velocity vector of ω at $\omega(t)$. When the manifold is $\mathbb{R}^n$, we will distinguish between the velocity vector and the derivative of the vector valued function, when necessary.

Unless otherwise stated, either explicitly or by unequivocal context, all our manifolds are C^∞, Hausdorff, with countable base, and connected. Unless otherwise indicated, *differentiable* means C^∞. When speaking of manifolds that possess boundary, our use of the word "manifold" (nearly) always refers to the interior. In particular, our compact manifolds are without boundary.

Let M be an n–dimensional differentiable manifold, with tangent bundle TM and associated natural projection $\pi : TM \to M$. For any $p \in M$, we let M_p denote the tangent space to M at p. We denote the collection of C^ℓ, $\ell = 0, 1, \ldots, \infty$, vector fields on M by $\Gamma^\ell(TM)$.

If $\phi : M \to N$ is a differentiable map from the manifold M to the manifold N, we let $\phi_* : TM \to TN$ denote the induced bundle map (in local coordinates the Jacobian linear transformation) linear on each fiber. We also let ϕ^* denote the pullback maps of the associated cotangent bundles.

Definition. A *connection on* M is a map $\nabla : TM \times \Gamma^1(TM) \to TM$, which we write as $\nabla_\xi Y$ instead of $\nabla(\xi, Y)$, with the following properties: First we require that $\nabla_\xi Y$ be in the same tangent space as ξ, and that for $\alpha, \beta \in \mathbb{R}$, $p \in M$, $\xi, \eta \in M_p$, $Y \in \Gamma^1(TM)$,

$$\nabla_{\alpha\xi+\beta\eta} Y = \alpha\nabla_\xi Y + \beta\nabla_\eta Y.$$

Second, we require that for $p \in M, \xi \in M_p, Y, Y_1, Y_2 \in \Gamma^1(TM)$, $f \in C^1(M)$, we shall have

$$\nabla_\xi(Y_1 + Y_2) = \nabla_\xi Y_1 + \nabla_\xi Y_2,$$
$$\nabla_\xi(fY) = (\xi f)Y_{|p} + f(p)\nabla_\xi Y.$$

Finally, we require that ∇ be smooth in the following sense: if $X, Y \in \Gamma^\infty(TM)$, then $\nabla_X Y \in \Gamma^\infty(TM)$.

The example that motivates the above definition is, naturally, $\mathbb{R}^n$. We let $\Im_p : \mathbb{R}^n \to (\mathbb{R}^n)_p$ be the natural identification of $\mathbb{R}^n$ with the (abstract) tangent space to $\mathbb{R}^n$ at any $p \in \mathbb{R}^n$. For the natural basis $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_n\}$ of $\mathbb{R}^n$, the natural basis of $(\mathbb{R}^n)_p$ determined by the chart consisting of the identity map of $\mathbb{R}^n$ to itself is given by

$$\partial_{j|p} = \Im_p \mathfrak{e}_j$$

for $j = 1, \ldots, n$. Let Y be a differentiable vector field on $\mathbb{R}^n$ such that $Y = \Sigma_j \eta^j \partial_j$. Then, the *standard connection on* $\mathbb{R}^n$ is given by

$$\text{(I.1.1)} \qquad \nabla_\xi Y = \sum_{j=1}^{n} (\xi \eta^j)\partial_j.$$

One easily checks that the requirements of the definition of a connection are satisfied.

A more explicit geometric expression for the standard connection on $\mathbb{R}^n$ is given as follows: Given $\xi \in (\mathbb{R}^n)_p$, let $\omega : (-\epsilon, \epsilon) \to \mathbb{R}^n \in C^1$ be a path in $\mathbb{R}^n$ with $\omega(0) = p$, $\omega'(0) = \xi$. Then one verifies that

$$\text{(I.1.2)} \qquad \nabla_\xi Y = \lim_{t \to 0} \frac{\Im_p \circ \Im_{\omega(t)}{}^{-1} Y_{|\omega(t)} - Y_{|p}}{t}.$$

Thus, the natural identification of the tangent spaces $(\mathbb{R}^n)_p$ and $(\mathbb{R}^n)_q$ via the map $\Im_q \circ \Im_p{}^{-1}$, for any p, q in $\mathbb{R}^n$, is that which allows for the natural differentiation of vector fields on $\mathbb{R}^n$. In an abstract differentiable manifold, no such natural identification exists, a priori. Therefore, it must be postulated in advance. However, it is far more natural to postulate the differentiation of vector fields first, and to then investigate the resultant identification of tangent spaces at different points of the manifold. See §I.2.

Let M be our differentiable manifold with connection ∇. We note that, for $p \in M, \xi \in M_p$, $\nabla_\xi Y$ is uniquely determined by the restriction of Y to any open set U containing p. To see this, fix $p \in M, \xi \in M_p$, and an open neighborhood U of p.

We first show that $Y|U = 0$ implies $\nabla_\xi Y = 0$. Pick a differentiable function $f : M \to \mathbb{R}$ such that $f(p) = 0$ and $f|M \setminus U = 1$. Then, $fY = Y$ and

$$\nabla_\xi Y = \nabla_\xi (fY) = (\xi f)Y_{|p} + f(p)\nabla_\xi Y,$$

both terms of which vanish at p. We conclude that, if two vector fields agree on all of U, then so do their covariant derivatives.

We may proceed conversely, namely, if Y is given as defined only on U, then pick open V relatively compact in U and $\phi : M \to [0, 1]$ differentiable with compact support such that $\phi|V = 1$ and $\operatorname{supp} \phi \subseteq U$. Then define $\overline{Y} \in \Gamma^1(TM)$ by setting $\overline{Y} = \phi Y$ on U, and $\overline{Y} = 0$ on $M \setminus U$; and finally define

$$\nabla_\xi Y = \nabla_\xi \overline{Y}.$$

Then, $\nabla_\xi Y$ is well-defined, that is, it is independent of the choice of extension of Y to $\overline{Y} \in \Gamma^1(TM)$. Thus, we may effectively calculate ∇ by restricting the vector fields in question to, for example, the domain of a chart.

We now show more, namely, that to calculate $\nabla_\xi Y$, for given $Y \in \Gamma^1(TM)$, we need only know Y restricted to a path through $p = \pi(\xi)$ with velocity vector at p equal to ξ. Indeed, let $x : U \to \mathbb{R}^n$ be a chart about p, and ξ given by

$$\xi = \sum_j \xi^j \partial_{j|p}.$$

Then $\nabla_\xi Y$ is given by

$$\nabla_\xi Y = \sum_j \xi^j \nabla_{\partial_{j|p}} Y.$$

Also, one has the functions $\eta^j : U \to \mathbb{R}$, $j = 1, \ldots, n$, such that

$$Y|U = \sum_j \eta^j \partial_j.$$

Now, there exist functions $\Gamma_{jk}{}^\ell : U \to \mathbb{R}$, $j, k, \ell = 1, \ldots, n$, referred to as *Christoffel symbols*, such that

$$\nabla_{\partial_k} \partial_j = \sum_\ell \Gamma_{jk}{}^\ell \partial_\ell \tag{I.1.3}$$

on U. We then have

$$\begin{aligned}
\nabla_\xi Y &= \sum_k \xi^k \nabla_{\partial_{k|p}} Y \\
&= \sum_k \xi^k \nabla_{\partial_{k|p}} \left(\sum_j \eta^j \partial_j \right) \\
&= \sum_k \xi^k \left\{ \sum_j (\partial_k \eta^j)(p) \partial_{j|p} + \sum_{j,\ell} \eta^j(p) \Gamma_{jk}{}^\ell(p) \partial_{\ell|p} \right\} \\
&= \sum_\ell \left\{ \sum_k \xi^k (\partial_k \eta^\ell)(p) + \sum_{j,k} \Gamma_{jk}{}^\ell(p) \eta^j(p) \xi^k \right\} \partial_{\ell|p},
\end{aligned}$$

that is,

$$\nabla_\xi Y = \sum_\ell \left\{ \sum_k \xi^k (\partial_k \eta^\ell)(p) + \sum_{j,k} \Gamma_{jk}{}^\ell(p) \eta^j(p) \xi^k \right\} \partial_{\ell|p}. \tag{I.1.4}$$

In particular, if $\omega : (\alpha, \beta) \to M$ is differentiable such that $t_0 \in (\alpha, \beta)$, $\omega(t_0) = p$, $\omega'(t_0) = \xi$, one then has

$$\nabla_\xi Y = \sum_\ell \left\{ (\eta^\ell \circ \omega)'(t_0) + \sum_{j,k} \Gamma_{jk}{}^\ell(p)\eta^j(p)\xi^k \right\} \partial_{\ell|p},$$

which was our claim.

Next, we note that the choice of connection on M is highly undetermined. Given any chart $x : U \to \mathbb{R}^n$, then any choice of n^3 functions $\Gamma_{jk}{}^\ell : U \to \mathbb{R} \in C^\infty$ determine a local connection on U, via the equations (I.1.3) and (I.1.4). One can then create global connections on M from local ones, by using a partition of unity.

Finally, we note the change of variable formula for the Christoffel symbols. Given two charts $x : U \to \mathbb{R}^n$, $y : U \to \mathbb{R}^n$, on M, with respective Christoffel symbols ${}_x\Gamma_{jk}{}^\ell$, ${}_y\Gamma_{st}{}^r$, then one verifies by direct calculation

(I.1.5)
$$\sum_\ell {}_x\Gamma_{jk}{}^\ell \frac{\partial(y^r \circ x^{-1})}{\partial x^\ell} = \frac{\partial^2(y^r \circ x^{-1})}{\partial x^j \partial x^k} + \sum_{s,t} \frac{\partial(y^s \circ x^{-1})}{\partial x^j} \frac{\partial(y^t \circ x^{-1})}{\partial x^k} {}_y\Gamma_{st}{}^r.$$

Definition. Let $\omega : (\alpha, \beta) \to M$ be a C^1 path in M. We define a *vector field along the path* ω to be a map $X : (\alpha, \beta) \to TM$, such that $\pi \circ X = \omega$, that is, $X(t) \in M_{\omega(t)}$ for all t. (Note that in such a situation, we do not necessarily obtain a vector field on the image of ω in M since it is possible, for example, that $t_1, t_2 \in (\alpha, \beta)$, $t_1 \neq t_2$, $\omega(t_1) = \omega(t_2)$, but $X(t_1) \neq X(t_2)$.)

We define the *derivative of X along ω*, $\nabla_t X$, as follows: Assume $x : U \to \mathbb{R}^n$ is a chart containing $\omega((\alpha, \beta))$ and define $\Gamma_{jk}{}^\ell$ as in (I.1.3). Also set

$$\omega^j = x^j \circ \omega, \quad j = 1, \ldots, n,$$

write X as

$$X = \sum_j \xi^j (\partial_j \circ \omega),$$

and finally, define

(I.1.6)
$$\nabla_t X = \sum_\ell \left\{ (\xi^\ell)' + \sum_{j,k} (\Gamma_{jk}{}^\ell \circ \omega)\xi^j(\omega^k)' \right\} (\partial_\ell \circ \omega),$$

for $X \in C^1$.

One checks, using (I.1.5), that the definition (I.1.6) is independent of the choice of chart on M and thereby obtains a well-defined vector field $\nabla_t X$ along ω even if the image of ω is not contained in the domain of one chart on M. Also,

$$\nabla_t(X_1 + X_2) = \nabla_t X_1 + \nabla_t X_2, \tag{I.1.7}$$

$$\nabla_t(fX) = f'X + f\nabla_t X, \tag{I.1.8}$$

for all vector fields X, X_1, X_2 along ω, and $f : (\alpha, \beta) \to \mathbb{R} \in C^1$.

One can now use the above to consider a more general situation, namely,

Definition. Let N, M be differentiable manifolds, $\phi : N \to M$ differentiable. Then, define a *vector field X along ϕ* to be a map $X : N \to TM$ satisfying $\pi \circ X = \phi$, that is, $X(q) \in M_{\phi(q)}$ for all $q \in N$.

If X is a differentiable vector field along ϕ, $q \in N$, $\xi \in N_q$, and ∇ a connection on M, define the *derivative of X along ϕ in the direction ξ*, $\nabla_\xi X$, as follows: Let $\omega : (-\epsilon, \epsilon) \to N$ be any differentiable path for which $\omega(0) = q$, $\omega'(0) = \xi$, and let $Y : (-\epsilon, \epsilon) \to TM$ be the vector field along $\phi \circ \omega$ given by

$$Y = X \circ \omega.$$

Define $\nabla_\xi X$ by

$$\nabla_\xi X = (\nabla_t Y)(0).$$

$\nabla_\xi X$ is seen to be independent of the choice of ω, and is therefore well-defined and satisfies

$$\nabla_\xi(X_1 + X_2) = \nabla_\xi X_1 + \nabla_\xi X_2,$$
$$\nabla_\xi(fX) = (\xi f)X_{|p} + f(p)\nabla_\xi X,$$

where X, X_1, X_2 are differentiable vector fields along ϕ and $f : N \to \mathbb{R}$ is differentiable.

§I.2. Parallel Translation of Vector Fields

Let M be a given differentiable manifold with connection ∇.

Definition. Let $\omega : (\alpha, \beta) \to M$ be a C^1 path in M. We say that a vector field X along ω is *parallel along ω* if

$$\nabla_t X = 0$$

on all of (α, β).

By (I.1.7), (I.1.8) one has that, given ω, ∇_t is a linear operator on vector fields along ω; thus, the set of parallel vector fields along ω is a vector space over $\mathbb{R}$. From (I.1.6), one has (via the theory of linear ordinary differential equations), to each $t_0 \in (\alpha, \beta)$, $\xi \in M_{\omega(t_0)}$, the existence of a unique parallel vector field X along ω satisfying $X(t_0) = \xi$. In particular, the space of parallel vector fields along ω is finite dimensional and has dimension equal to that of M.

Thus, we can construct isomorphisms between the tangent spaces to M at different points of ω, namely, let $t_1, t_2 \in (\alpha, \beta)$, and for $\xi \in M_{\omega(t_1)}$ let X_ξ be the parallel vector field along ω satisfying $X_\xi(t_1) = \xi$. Now set

$$\tau_{t_1,t_2}(\xi) = X_\xi(t_2).$$

Then, τ_{t_1,t_2} is a linear isomorphism of $M_{\omega(t_1)}$ onto $M_{\omega(t_2)}$ and is called *parallel translation along* ω *from* $M_{\omega(t_1)}$ *to* $M_{\omega(t_2)}$.

Theorem I.2.1. *Let* $\omega : (\alpha, \beta) \to M$ *be a differentiable path, X a differentiable vector field along* ω*, and* $t_0 \in (\alpha, \beta)$*. Then,*

$$(\nabla_t X)(t_0) = \lim_{t \to t_0} \frac{\tau_{t,t_0}(X(t)) - X(t_0)}{t - t_0}. \tag{I.2.1}$$

Proof. Let $E_1(t), \ldots, E_n(t)$ be n parallel vector fields along ω, which are pointwise linearly independent (of course, as soon as they are linearly independent at one point, they are linearly independent at all points), $n = \dim M$; then, there exist functions $\xi^j : (\alpha, \beta) \to \mathbb{R}$, $j = 1, \ldots, n$ such that

$$X(t) = \sum_{j=1}^{n} \xi^j(t) E_j(t)$$

on (α, β). One now calculates explicitly both sides of (I.2.1) and the result follows. ■

Remark I.2.1. The reader is invited to compare (I.2.1) with (I.1.2). Note that the identification of tangent spaces $\tau_{t_1,t_2}(\xi)$ depends on the path ω connecting $\omega(t_1)$ to $\omega(t_2)$. A local calculation shows that if parallel translation of vector fields is independent of the choice of path connecting any two given points in M, then the curvature tensor of ∇ – to be defined below in §I.4 – vanishes identically on M. Almost needless to say, if parallel translation of vector fields on M were independent of the choice of path connecting any two given points in M, then one could construct, at will, n linearly independent (over $\mathbb{R}$) nonvanishing vector fields on M – a global *topological* restriction on M. See, also, Remark I.5.2.

§I.3. Geodesics and the Exponential Map

We are still with our differentiable manifold M and connection ∇.

Definition. A path $\omega : (\alpha, \beta) \to M \in C^\ell$, $\ell \geq 2$, is called is called a *geodesic* if

$$\nabla_t \omega' = 0. \tag{I.3.1}$$

on all of (α, β).

To write the equation for a geodesic in a chart, we let $x : U \to \mathbb{R}^n$ be the chart, set $\omega^j = x^j \circ \omega$, $j = 1, \ldots, n$, and let $\Gamma_{jk}{}^\ell$, $j, k, \ell = 1, \ldots, n$ be given by (I.1.3). Then, (I.1.6) implies that (I.3.1) reads as

$$(\omega^\ell)'' + \sum_{j,k} (\Gamma_{jk}{}^\ell \circ \omega)(\omega^j)'(\omega^k)' = 0. \tag{I.3.2}$$

We now exhibit the second-order system (I.3.2) as a first-order system on TM. With the projection $\pi : TM \to M$ and chart $x : U \to \mathbb{R}^n$, we associate the natural chart $Q : \pi^{-1}[U] \to \mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ by $Q(\xi) = (q(\xi), \dot{q}(\xi))$, where

$$q = x \circ \pi, \qquad \dot{q}(\xi) = \xi x$$

(where by ξx we mean $\xi^j = \xi x^j$, $j = 1, \ldots, n$). Thus,

$$\xi = \sum_j \dot{q}^j(\xi) \partial_{j|\pi(\xi)}.$$

We find it convenient to write the basis of tangent spaces to TM at points of $\pi^{-1}[U]$ by $\{\partial/\partial q^1, \ldots, \partial/\partial q^n, \partial/\partial \dot{q}^1, \ldots, \partial/\partial \dot{q}^n\}$. One immediately has

$$\pi_*(\partial/\partial q^j) = \partial/\partial x^j, \qquad \pi_*(\partial/\partial \dot{q}^j) = 0.$$

The differential equation (I.3.2) can then be written as a first-order equation in $\pi^{-1}[U]$:

$$(q^\ell)' = \dot{q}^\ell, \tag{I.3.3}$$

$$(\dot{q}^\ell)' = -\sum_{j,k} (\Gamma_{jk}{}^\ell \circ \pi) \dot{q}^j \dot{q}^k. \tag{I.3.4}$$

The solutions to (I.3.3), (I.3.4) are therefore integral curves of the vector field $\mathcal{G}$ on $\pi^{-1}[U]$ given by

$$\mathcal{G} = \sum_\ell \left\{ \dot{q}^\ell \frac{\partial}{\partial q^\ell} - \sum_{j,k} (\Gamma_{jk}{}^\ell \circ \pi) \dot{q}^j \dot{q}^k \frac{\partial}{\partial \dot{q}^\ell} \right\}.$$

Since the geodesic equations are independent of the choice of coordinates on M, we conclude that $\mathcal{G}$ defines a global vector field on TM.

Definition. The maximal flow of $\mathcal{G}$ is called the *geodesic flow*.

One easily has the following:

Theorem I.3.1. *Let $\Omega : (\alpha, \beta) \to TM$ be an integral curve of $\mathcal{G}$, and $\omega = \pi \circ \Omega$. Then,*

(I.3.5) $$\omega' = \Omega$$

and ω is a geodesic in M. Conversely, given a geodesic $\omega : (\alpha, \beta) \to M$ and Ω defined by (I.3.5)*, then Ω is an integral curve of $\mathcal{G}$.*

Thus, if $\varphi(t, \xi)$ denotes the maximal flow of $\mathcal{G}$ on TM, where $t \in \mathbb{R}$, $\xi \in TM$, then

$$\gamma_\xi(t) := \pi \circ \varphi(t, \xi)$$

is the unique maximal (*relative to its domain in* $\mathbb{R}$) *geodesic in M satisfying*

$$\gamma_\xi(0) = \pi(\xi), \quad {\gamma_\xi}'(0) = \xi.$$

Of course,

$${\gamma_\xi}'(t) = \varphi(t, \xi).$$

In particular, $\gamma_\xi(t)$ depends differentiably (*i.e.*, C^∞) *on t and ξ.*

Finally, if I_ξ is the maximal interval on which γ_ξ is defined, then for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$ we have

(I.3.6) $$I_{\alpha\xi} = (1/\alpha)I_\xi, \quad \gamma_{\alpha\xi}(t) = \gamma_\xi(\alpha t),$$

where if $I_\xi = (\beta_1, \beta_2)$ then $(1/\alpha)I_\xi := (\beta_1/\alpha, \beta_2/\alpha)$ when $\alpha > 0$, and $(1/\alpha)I_\xi := (\beta_2/\alpha, \beta_1/\alpha)$ when $\alpha < 0$.

Remark I.3.1. We note that (I.3.5) – the coordinate-free version of (I.3.3), (I.3.4) – is the heart of a coordinate-free definition of a second-order differential equation on a manifold. Namely, we say that *a vector field $\mathfrak{X}$ on TM determines a second-order ordinary differential equation on M* if

$$\pi_*(\mathfrak{X}_{|\xi}) = \xi$$