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Drilling short geodesics in hyperbolic 3-manifolds

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Abstract

We give an expository account of the deformation theory of geometrically finite, 3-dimensional hyperbolic cone-manifolds and its application to three classical conjectures about Kleinian groups.

1. Introduction

In a series of papers ([HK98, HK02, HK]), Hodgson and Kerckhoff developed a deformation theory for 3-dimensional hyperbolic cone-manifolds which they used to prove various important results about closed and finite volume hyperbolic 3-manifolds. This deformation theory was extended to infinite volume, geometrically finite hyperbolic cone-manifolds in [Bro04b, Bro04a]. In this setting the deformation theory has had a number of applications to classical conjectures about Kleinian groups.

Here is an example of a basic problem that can be addressed via the deformation theory. Let (M,g) be a geometrically finite hyperbolic 3-manifold that contains a simple closed geodesic γ. Let $\hat{M} = M \backslash \gamma$ be the complement of γ. There will be then be a unique, geometrically finite, complete hyperbolic metric $\hat{g}$ on $\hat{M}$ such that the conformal boundaries of (M,g) and $(\hat{M},\hat{g})$ agree. We have the following theorem

Theorem 1.1 ([BB04])**.** *For each $K > 1$ there exists an $\ell > 0$ such that if the length of γ in (M,g) is less then ℓ then there exists a K-bi-Lipschitz map*

$$\phi : (M \backslash \mathbb{T}, g) \longrightarrow (\hat{M} \backslash \hat{\mathbb{T}}, \hat{g})$$

where $\mathbb{T}$ and $\hat{\mathbb{T}}$ are Margulis tubes about γ and the rank two cusp, respectively.

We call such a theorem a "drilling theorem" for we have drilled the geodesic γ out of the hyperbolic manifold (M,g).

The way we obtain geometric control of the metric $\hat{g}$ is to interpolate between g and $\hat{g}$ using *hyperbolic cone-metrics*. The Hodgson-Kerckhoff deformation theory gives means to bound the change in geometry as this one-parameter family of metrics

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varies. The first part of this paper will be an exposition of this deformation theory emphasizing the most geometric parts. For an expository account of Hodgson and Kerckhoff's work see [HK03]. To keep this paper somewhat self-contained there is some necessary overlap between the two papers.

In the second part of the paper we will apply the deformation theory to a collection of classical conjectures in Kleinian groups: the density conjecture, density of cusps on the boundary of quasiconformal deformation spaces and the ending lamination conjecture. Rather than discussing these conjectures in their full generality we will restrict to the special case of a Bers' slice. This will allow us to demonstrate how the deformation theory plays a role in approaching the conjectures in a simpler setting.

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2. Deformations of hyperbolic metrics

We will begin by examing the various different ways one can study a family of hyperbolic metrics: as Riemannian metrics, as (G,X)-structures and as representations of the fundamental group in the space of hyperbolic isometries. We will see the advantages of each viewpoint and the connections between the different viewpoints. A reference for this material is §1 and §2 of [HK98].

In the final subsection we will discuss complex projective structures on surfaces. These arise naturally as the boundary of hyperbolic 3-manifolds and will play an important role in the extension of the Hodgson-Kerckhoff deformation theory to infinite volume and geometrically finite hyperbolic cone-manifolds.

2.1. One-parameter families of metrics

We start with a family of metrics, $g_t : V \times V \longrightarrow R$, on a finite dimensional vector space V. For each t there is a unique $\eta_t \in \hom(V,V)$ such that

$$\frac{dg_t(v,w)}{dt} = 2g_t(v,\eta_t(w)). \tag{2.1}$$

Since g_t is symmetric, η_t is self-adjoint, i.e.

$$g_t(\eta_t(v),w) = g_t(v,\eta_t(w)).$$

We measure the size of η_t using the metric g_t. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V in the g_t metric. Then define the norm of η_t by the formula

$$\|\eta_t\|^2 = \sum g_t(\eta_t(e_i), \eta_t(e_i)). \tag{2.2}$$

For any $v \in V$ we then have

$$g_t(v, \eta_t(v)) \leq \|\eta_t\| g_t(v, v).$$

By integrating (2.1) we see that if $\|\eta_t\| \leq K$ for all $t \in [0, T]$ then

$$e^{-2KT} g_0(v, v) \leq g_T(v, v) \leq e^{2KT} g_0(v, v).$$

In particular the identity map on V is a KT-bi-Lipschitz map from the g_0-metric to g_T-metric.

The trace of η_t is the *divergence* and it is the derivative of the volume. The traceless part of η_t is the *strain* and it measures the change in the conformal structure.

2.2. Metrics on a manifold

Now we apply the above work to a family of metrics, g_t, on a differentiable manifold M. In this setting η_t is a one-parameter family in $\hom(TM, TM)$. Let $\|\eta_t(p)\|$ be the pointwise norm of η_t. Let $\phi_t : (M, g_0) \longrightarrow (M, g_t)$ be the identity map on M. If $\|\eta_t(p)\| \leq K$ for all $p \in M$ and all $t \in [0, T]$ then ϕ_t is a KT-bi-Lipschitz diffeomorphism.

The identity map on M may not have the smallest bi-Lipschitz constant of all maps from (M, g_0) to (M, g_t). In particular for an arbitrary family of metrics there is no reason to hope that we can control the norm of η_t. The driving idea behind the Hodgson-Kerckhoff deformation theory is to find one-parameter families of hyperbolic metrics g_t where the derivative η_t is a *harmonic strain field*. As we will see below, this extra structure will allow us to control the norm of η_t.

2.3. Hyperbolic metrics on a manifold

Let $\mathcal{H}(M)$ be the space of all hyperbolic metrics on M. Two metrics g and h in $\mathcal{H}(M)$ are equivalent if there is a diffeomorphism $\psi : M \longrightarrow M$ isotopic to the identity such that $h = \psi^* g$. Given two equivalence classes of metrics we want to find an efficient path between them. That is we want to find a path g_t that minimizes the derivative η_t. The last statement can be interpreted in a number of ways. For example, we could try to minimize the pointwise or L^2-norm of η_t. However, if M is not compact then both of these norms can and will be infinite. Our efficient paths will have two properties. First, they will be divergence free so that η_t is a strain field. Second they will be harmonic. We will not formally define harmonic. Informally, one can think of a harmonic strain field as locally minimizing the L^2-norm (see Appendix B of [McM96]).

A harmonic strain field satisfies the following important equation:

Theorem 2.1. *Let (M,g) be a compact hyperbolic manifold with boundary and let η be a harmonic strain field. Then*

$$\int_M \|\eta\|^2 + \|\nabla\eta\|^2 = \int_{\partial M} *\nabla\eta\wedge\eta. \tag{2.3}$$

This formula is very important because it allows us to compute the L^2-norm of a strain field by only knowing information on the boundary. We also note that η is harmonic if it satisfies (2.3) for all compact submanifolds.

Another feature of harmonic strain fields is that they satisfy a mean value inequality:

Theorem 2.2. *Let (M,g) be a hyperbolic manifold and η a harmonic strain field. If B is a ball in M of radius $R > \frac{\pi}{\sqrt{2}}$ centered at p then*

$$\|\eta(p)\| \leq \frac{3\sqrt{2(B)}}{4\pi f(R)}\sqrt{\int_B \|\eta\|^2 dV}$$

where $f(R) = \cosh(R)\sin(\sqrt{2}R) - \sqrt{2}\sinh(R)\cos(\sqrt{2}r)$.

Together, Theorems 2.1 and 2.2 will allow us to get pointwise bounds on the the norm of η, at least for points in the thick part of (M,g).

2.4. Developing maps

Another way to think of a hyperbolic structure is as a (G,X)-structure, where X is hyperbolic space and G the group of hyperbolic isometries. A (G,X) structure is an atlas of charts to X with transition maps which are restrictions of elements of G. A (G,X)-structure determines a developing map and a holonomy representation.

Here's how it works for a hyperbolic 3-manifold: A *developing map* is a local diffeomorphism,

$$D : \tilde{M} \longrightarrow \mathbb{H}^3,$$

and the *holonomy representation* is a representation of the fundamental group,

$$\rho : \pi_1(M) \longrightarrow PSL_2\mathbb{C} = \mathrm{Isom}^+(\mathbb{H}^3).$$

The developing map commutes with the action of the fundamental group where the fundamental groups acts on $\tilde{M}$ as deck transformations and on $\mathbb{H}^3$ via the holonomy representation. That is

$$D(\gamma(x)) = \rho(\gamma)D(x) \tag{2.4}$$

for all $\gamma \in \pi_1(M)$. Let $\tilde{g}$ be the pull back of the hyperbolic metric. Then (2.4) implies that $\tilde{g}$ is equivariant and descends to a hyperbolic metric g on M.

Conversely, a hyperbolic manifold, (M,g), determines a developing map and holonomy representation. The developing map is unique up to post-composition with hyperbolic isometries. If we post-compose the developing with an isometry $\alpha \in PSL_2\mathbb{C}$ then we conjugate the holonomy by α.

Given a smooth family of hyperbolic metrics (M, g_t), there is a smooth family of developing maps D_t, and holonomy representations ρ_t. The derivative of the developing maps determines a family of vector fields v_t on $\tilde{M}$ in the following way. For a point $x \in \tilde{M}$, $D_t(x)$ is smooth path in $\mathbb{H}^3$. Let $v_t(x)$ be the pull-back, via D_t, of the tangent vector of this path at time t. These vector fields are not equivariant. However, they do satisfy the following *automorphic* property. For all $\gamma \in \pi_1(M)$ the difference, $\gamma_* v_t - v_t$, is an infinitesimal isometry in the $\tilde{g}_t$-metric. That is, the flow of the vector field $\gamma_* v_t - v_t$ is an isometry. This follows directly from differentiating (2.4).

The automorphic vector fields v_t, lead to the connection between the developing maps and the derivative, η_t, of the metrics g_t. The covariant derivative, $\nabla_t v_t$, is an element of $\hom(T\tilde{M}, T\tilde{M})$. Let $\operatorname{sym}\nabla_t v_t$ be its symmetric part. The covariant derivative of an infinitesimal isometry is skew. Therefore, the automorphic property of v_t implies that $\operatorname{sym}\nabla_t v_t$ is equivariant and descends to an element of $\hom(TM, TM)$. By noting that the derivative $\frac{dg_t(v,w)}{dt}$ is the Lie derivative $\mathcal{L}_{v_t} g_t(v,w)$ we see that $\operatorname{sym}\nabla_t v_t = \eta_t$.

2.5. Holonomy representations

Let $\mathcal{R}(M)$ be the space of representations of $\pi_1(M)$ in $PSL_2\mathbb{C}$. We are only interested in representations up to conjugacy so we would like to study the quotient of $\mathcal{R}(M)$ under the action of $PSL_2\mathbb{C}$ by conjugacy. Unfortunately, this quotient may not be a nice object. For instance it may not even by Hausdorff. Instead one takes the *Mumford quotient* of $\mathcal{R}(M)$ which we denote $R(M)$. The Mumford quotient is an algebraic variety and its Zariski tangent space at a representation ρ is the cohomology group $H^1(\pi_1(M); \operatorname{Ad}\rho)$. It will turn out, that at all points were are interested in, $R(M)$ is simply the topological quotient of $\mathcal{R}(M)$ by conjugacy. Furthermore, at these points $R(M)$ will be a differentiable manifold and the the Zariski tangent space will be naturally identified with the differentiable tangent space. For this reason we will ignore the distinction between the Mumford quotient and the topological quotient.

By differentiating a smooth family of representations ρ_t we can see how the differentiable tangent space at each ρ_t is identified with $H^1(\pi_1(M); \mathrm{Ad}\,\rho_t)$. Let γ be an element of $\pi_1(M)$. Then $\rho_t(\gamma)$ is a smooth path in $PSL_2\mathbb{C}$. Each tangent space of $PSL_2\mathbb{C}$ is canonically identified with the Lie algebra $sl_2\mathbb{C}$. Therefore the derivative $\dot{\rho}_t$ can be thought of as a map

$$\dot{\rho}_t : \pi_1(M) \longrightarrow sl_2\mathbb{C}$$

for each t. This map satisfies the cocyle condition

$$\dot{\rho}_t(\gamma\beta) = \dot{\rho}_t(\gamma) + \mathrm{Ad}\,\rho_t(\gamma)\dot{\rho}_t(\beta)$$

for all γ and β in $\pi_1(M)$ and therefore determines a cohomology class in $H^1(\pi_1(M); \mathrm{Ad}\,\rho_t)$.

We also remark that $\dot{\rho}_t(\gamma)$ corresponds to the vector field $\gamma_* v_t - v_t$. The latter vector field is identified with an element of $sl_2\mathbb{C}$ by pushing forward $\gamma_* v_t - v_t$ via D_t. This push foward is an infinitesimal isometry on $\mathbb{H}^3$ and the space of infinitesimal isometries of $\mathbb{H}^3$ is canonically identified with $sl_2\mathbb{C}$.

2.6. Complex projective structures

A *complex projective structure* on a surface S is an atlas of charts to the Riemann sphere, $\widehat{\mathbb{C}}$, where the transition maps are restrictions of elements of $PSL_2\mathbb{C}$. A projective structure is another example of (G,X)-structure where $G = PSL_2\mathbb{C}$ and $X = \widehat{\mathbb{C}}$. Let $P(S)$ be the space of projective structures on S. Since the action of $PSL_2\mathbb{C}$ is conformal, a projective structure also determines a conformal structure on S so there is a map

$$P(S) \longrightarrow T(S)$$

where $T(S)$ is the *Teichmüller space* of marked conformal structures on S. One is often interested in the space of projective structures with a fixed conformal structure X. We denote the space of such structures $P(X)$.

Elements of $PSL_2\mathbb{C}$ take round circles in $\widehat{\mathbb{C}}$ to round circles. Therefore, there is a well defined notion of a round circle on a projective structure. A conformal map f between two projective structures Σ and Σ' will distort these round circles. The *Schwarzian derivative*, Sf, measures this distortion. We will not give an exact definition of Sf although we will describe an infinitesimal version below. We will however state the key properties of the Schwarzian derivative that we will use. First, Sf is a holomoprhic quadratic differential on X. The quotient of the absolute value of a holomorphic quadratic differential and a metric is a function. Using the unique hyperbolic metric on X we can take the sup-norm of this function to a define the sup-norm, $\|Sf\|_\infty$, of the Schwarzian. This determines a metric on $P(X)$ by setting $d(\Sigma, \Sigma') = \|Sf\|_\infty$. Furthermore, given any holomorphic quadratic differential Φ on X there is a projective structure Σ' such that for the conformal map $f : \Sigma \longrightarrow \Sigma'$, $Sf = \Phi$. Therefore $P(X)$

is isomorphic to the vector space $Q(X)$ of holomorphic quadratic differentials on X.

A projective structure is *Fuchsian* if it is the quotient of a round disk in $\widehat{\mathbb{C}}$. There is a unique Fuchsian projective structure, Σ_F, in each $P(X)$. We will often be interested in the distance between an arbitrary projective structure $\Sigma \in P(X)$ and this unique Fuchsian projective structure. We therefore let $\|\Sigma\|_F = d(\Sigma, \Sigma_F)$.

As with any (G,X)-structure, a projective structure Σ on S determines a developing map

$$D : \tilde{S} \longrightarrow \widehat{\mathbb{C}}$$

and a holonomy representation

$$\rho : \pi_1(S) \longrightarrow PSL_2\mathbb{C}$$

satisfying (2.4). Now let Σ_t be a smooth path of projective structures in $P(X)$. Then there is a smooth path of developing maps D_t which determine vector fields v_t on $\tilde{S}$. The developing maps, D_t, can be chosen to be conformal maps from $\tilde{X}$ to $\widehat{\mathbb{C}}$ which will make the vectors fields v_t conformal on $\tilde{X}$.

Let $v(z)$ be a conformal vector field on a domain in $\widehat{\mathbb{C}}$. Then $v(z) = f(z)\frac{\partial}{\partial z}$ where f is a holomorphic function. A conformal vector field is *projective* if its flow consists of elements of $PSL_2\mathbb{C}$. The space of projective fields is the Lie algebra $sl_2\mathbb{C}$ and $v(z)$ will be projective if and only if $f(z)$ is a quadratic polynomial. At each point z in the domain let $s(z)$ be the unique projective vector field that best approximates v at z. Note that $s(z)$ is obtained by taking the first three terms of the Taylor series of f at z. Differentiating $s(z)$ we obtain an $sl_2\mathbb{C}$-valued 1-form which can be canonically associated with a holomorphic quadratic differential. This quadratic differential is the Schwarzian derivative, Sv, of the vector field v.

We now return to our path of projective structures Σ_t in $P(X)$. The Schwarzian derivative of the conformal vector fields v_t will be equivariant and therefore Sv_t will be a holomorphic quadratic differential on X. The norm $\|Sv_t\|_\infty$ is the infinitesimal version of the metric on $P(X)$ and if we can bound it for all t we bound the distance between Σ_0 and Σ_1.

We need one final fact about projective structures. The holonomy representation defines a map from $P(S)$ to the space $R(S)$ of representations of $\pi_1(S)$ in $PSL_2\mathbb{C}$ modulo conjugacy. We the have the following theorem.

Theorem 2.3 ([Hej75, Ear81, Hub81])**.** *The holonomy map*

$$\text{hol} : P(S) \longrightarrow R(S)$$

is a holomorphic, local homeomorphism.

3. Hyperbolic cone-manifolds

3.1. Geometrically finite hyperbolic cone-manifolds

Let N be a compact manifold with boundary, $\mathcal{C}$ a collection of simple closed curves in the interior of N and M the interior of $N \backslash \mathcal{C}$. Let g be a complete metric on the interior of N that is a smooth Riemannian metric on M. We say that g is a hyperbolic *cone-metric* if the following holds: First g is a hyperbolic metric on M. Second, for points on $\mathcal{C}$ the metric has the form

$$dr^2 + \sinh^2 r d\theta^2 + \cosh^2 r dz^2$$

where θ is measured modulo some *cone-angle* α. Note that the cone-angle must be locally constant on $\mathcal{C}$. Therefore there is a cone-angle associated to each component of $\mathcal{C}$.

Since the metric g is complete the boundary ∂N consists of tori and higher genus surfaces. Let $\partial_0 N$ denote the higher genus components of the boundary. To develop a good deformation theory we need to assume that there metric g has certain asymptotic behavior as we approach $\partial_0 N$. We say that a hyperbolic, cone-metric g is *geometrically finite* if the hyperbolic structure extends to a *projective structure* on $\partial_0 N$. More explicitly g is geometrically finite if for each $p \in \partial_0 N$ there exists an open neighborhood of p in N and a map $\psi : V \longrightarrow \mathbb{H}^3 \cup \widehat{\mathbb{C}}$ that is a homeomorphism onto its image and is an isometry on $V \cap \operatorname{int} M$. The restriction of ψ to $V \cap \partial_0 N$ will determine an atlas of charts to $\widehat{\mathbb{C}}$. Since hyperbolic isometries of $\mathbb{H}^3$ extend to projective transformations of $\widehat{\mathbb{C}}$ this atlas will determine a projective structure on $\partial_0 N$.

Let $GF(N, \mathcal{C})$ be equivalence classes of geometrically finite hyperbolic cone-manifolds on the pair $(N, \mathcal{C})$. If g is a hyperbolic cone-metric on $(N, \mathcal{C})$ we refer to the induced projective structure on $\partial_0 N$ as the *projective boundary*. The projective structure induces a conformal structure on $\partial_0 N$. This is the *conformal boundary*.

Note that the round circles in the projective boundary are the boundary at infinity of hyperbolic planes in the hyperbolic manifold. As the 3-dimensional hyperbolic metric deforms these planes will not stay totally geodesic. This will be detected by the change in the projective boundary.

3.2. Deformations of hyperbolic cone-manifolds

A *meridian* for the pair $(N, \mathcal{C})$ is a simple closed curve $\gamma \subset \operatorname{int} N$ that bounds a disk in N which intersects $\mathcal{C}$ in a single point. Each component of $\mathcal{C}$ has a unique meridian up to homotopy in $M = \operatorname{int} N \backslash \mathcal{C}$. Furthermore if ρ is the holonomy of a cone-manifold structure on $(N, \mathcal{C})$ then $\rho(\gamma)$ will be elliptic (or the identity if the cone angle is a multiple of 2π) for all meridians γ.

On the other hand there certainly will be representations where not all meridians are elliptic. For this reason we let $R_e(M)$ be the subset of $R(M)$ where the meridians are elliptic or the identity. We then have the following theorem which is essentialy due to Thurston ([Thu80]).

Theorem 3.1. *The holonomy map*

$$\text{hol} : GF(N, \mathcal{C}) \longrightarrow R_e(M)$$

is a local homeomorphism.

With this theorem our next goal is to give a local parameterization of $R(M)$. To do this we first need to define parameters. This local parameterization will be of a neighborhood in $R(M)$, not just a neighborhood in $R_e(M)$. These more general representations also have geometric signifigance. They correspond to Thurston's *generalized Dehn surgery singularities*. We will not explain the geometry of these singularities here.

Let

$$\mathcal{L}_{\mathcal{M}} : R(M) \longrightarrow \mathbb{C}^k$$

be the holomorphic map which assigns to each representation the k-tuple of complex lengths of the k-meridians of $(N, \mathcal{C})$. This is our first set of parameters.

The second set of parameters comes from the conformal boundary. Given a component S of $\partial_0 N$ we can define a map from $GF(N, \mathcal{C})$ to the Teichmüller space $T(S)$. This map assigns to each geometrically finite cone-manifold the conformal boundary structure on S. If $\rho \in R(M)$ is the holonomy of a cone-manifold in $GF(N, \mathcal{C})$ then by pre-composing this map with hol^{-1}, we obtain a map ∂_S from a neighborhood of ρ in $R_e(M)$ to $T(S)$. Here we choose the unique branch of hol^{-1} that takes ρ to the given geometrically finite cone-manifold. There is then a unique holomorphic extension of ∂_S to a neighborhood of ρ in $R(M)$.

Repeating the construction for each component of $\partial_0 N$ and combining the maps we have a single map

$$\partial : R(M) \longrightarrow T(\partial_0 N).$$

Strictly speaking ∂ is only defined for a neighborhood of ρ in $R(M)$. We also note that there are examples of distinct geometrically finite hyperbolic cone-manifolds with the same holonomy representation. When this happens each manifold will define a different boundary map ∂.

Now we combine our two parameters. Define

$$\Phi : R(M) \longrightarrow \mathbb{C}^k \times T(\partial_0 N)$$

by $\Phi(\rho) = (\mathcal{L}_{\mathcal{M}}(\rho), \partial(\rho))$.

Theorem 3.2 ([HK98, HK, Bro04b]). *Let* ρ *be the holonomy of a geometrically finite cone-manifold. If the cone-angle is* $\leq 2\pi$ *or the tube radius of the singular locus is* $\geq \sinh^{-1} 1/\sqrt{2}$ *then the map* Φ *is a holomorphic local homeomorphism.*

Sketch of proof of theorem 3.2. By a theorem of Thurston

$$\dim_{\mathbb{C}} R(M) \geq k + \dim_{\mathbb{C}} T(\partial_0 N).$$

Since the map Φ is holomorphic if we can show that the derivative, Φ_*, is injective at ρ then Φ will be a local homeomorphism at ρ.

The first step in proving this injectivity is a Hodge theorem: Any tangent vector of $R(M)$ at ρ that is in the kernel of ∂_* is represented by a harmonic strain field η on (M, g_α). Note there are some subtle issues to proving this Hodge theorem since our manifold is not compact and the metric is not complete. In particular, the harmonic strain field η is only unique after making some choice of boundary conditions for the solution.

Next we would like to calculate the L^2-norm of η on M. Theorem 2.1 tells how to calculate the L^2-norm of a harmonic strain field on a compact manifold with boundary. We can obtain a similar formula for harmonic strain fields on a geometrically finite manifold if the strain field fixes the conformal boundary. Analytically this is equivalent to $\partial_* \eta = 0$ where ∂_* is the tangent map of the boundary map ∂ from $R(M)$ to $T(\partial_0 N)$. The pointwise norm of such conformal deformations will decay exponentially and the boundary term in (2.3) will limit to zero for surfaces exiting the geometrically finite end. This allows us to calculate the L^2-norm of η even on the non-compact geometrically finite ends. In particular, we have

$$\int_{M \backslash U} \|\eta\|^2 + \|\nabla \eta\|^2 = \int_{\partial U} *\nabla \eta \wedge \eta$$

where U is tubular neighborhood of the singular locus, even though $M \backslash U$ is not compact. Note that in general the L^2-norm will be infinite on all of M.

The final step is to calculate the boundary term. This is done in the following way. In a tubular neighborhood of the singular locus we can decompose η as the sum of two strain fields, $\eta = \eta_0 + \eta_c$. The first term, η_0, is an explicit model deformation completely determined by the derivatives of the complex lengths of the components