PART I

FUNDAMENTAL CONCEPTS IN CONTINUUM MECHANICS

CHAPTER ONE

Describing the motion of a system: geometry and kinematics

1.1. Deformations

The purpose of mechanics is to study and describe the motion of material systems. The language of mechanics is very similar to that of set theory in mathematics: we are interested in material bodies or systems, which are made of material points or matter particles. A material system fills some part (a subset) of the ambient space ($\mathbb{R}^3$), and the position of a material point is given by a point in $\mathbb{R}^3$; a part of a material system is called a subsystem.

We will almost exclusively consider material bodies that fill a domain (i.e., a connected open set) of the space. We will not study the mechanically important cases of thin bodies that can be modeled as a surface (plates, shells) or as a line (beams, cables). The modeling of the motion of such systems necessitates hypotheses that are very similar to the ones we will present in this book, but we will not consider these cases here.

A material system fills a domain Ω_0 in $\mathbb{R}^3$ at a given time t_0. After deformation (think, for example, of a fluid or a tennis ball), the system fills a domain Ω in $\mathbb{R}^3$. A material point, whose initial position is given by the point $a \in \Omega_0$, will be, after transformation, at the point $x \in \Omega$.

The deformation can thus be characterized by a mapping as follows (see Figure 1.1):

$$\Phi: a \in \Omega_0 \mapsto x \in \Omega.$$

Assuming that matter is conserved during the deformation, we are led to make the following natural hypothesis:

$$\textit{The function } \Phi \textit{ is one – to – one from } \Omega_0 \textit{ onto } \Omega.$$

We will further assume that the deformation Φ is a smooth application of class $\mathcal{C}^1$ at least, from Ω_0 into Ω, as well as its inverse ($\mathcal{C}^1$ from Ω onto Ω_0). In fact we assume that Φ is as smooth as needed.

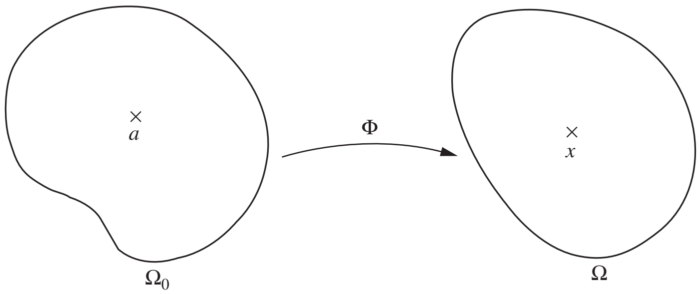

Figure 1.1 The mapping Φ.

Regularity assumption

The regularity assumption made on Φ will actually be general; we will assume that all the functions we introduce are as regular as needed for all the mathematical operations performed to be justified (e.g., integration by parts, differentiation of an integral depending on a parameter, etc.). This hypothesis, which will be constantly assumed in the following, will only be weakened in Chapter 6 for the study of shock waves, which correspond to the appearance of discontinuity surfaces. In that case, we will assume that the map Φ is piecewise $\mathcal{C}^1$. This assumption must be weakened also for the study of other phenomena which will not be considered here, such as singular vortices for fluids, dislocations for solids, or collisions of rigid bodies.

Let $\operatorname{grad} \Phi(a) = \nabla\Phi(a)$ be the matrix whose entries are the quantities $(\partial\Phi_i/\partial a_j)(a)$; this is the Jacobian matrix of the mapping $a \mapsto x$ also denoted sometimes Dx/Da. Because Φ^{-1} is differentiable, the Jacobian $\det(\nabla\Phi)$ of the transformation $a \mapsto x$ is necessarily different from zero. We will assume in the following that it is strictly positive; the negative sign corresponds to the nonphysical case of a change of orientation (a left glove becoming a right glove). We will later study the role played by the linear tangent map at point a in relation to the Taylor formula

$$\Phi(a) = \Phi(a_0) + \nabla\Phi(a_0)\cdot(a - a_0) + o(|a - a_0|).$$

We will also introduce the dilation tensor to study the deformation of a "small" tetrahedron.

Displacement

Definition 1.1. *The map* $u : a \mapsto x - a = \Phi(a) - a$ *is called the displacement;* $u(a)$ *is the displacement of the particle* a.

Elementary deformations

Our aim here is to describe some typical elementary deformations.

a) Rigid deformations

The displacement is called rigid (in this case, we should no longer talk about deformations) when the distance between any pair of points is conserved as follows:

$$d(a, a') = d(x, x'), \qquad \forall\, a, a' \in \Omega_0,$$

where $x = \Phi(a)$, $x' = \Phi(a')$. This is equivalent to assuming that

$$\Phi \textit{ is an isometry from } \Omega_0 \textit{ onto } \Omega,$$

or, when Ω_0 is not included in an affine subspace of dimension less than or equal to 2,

$$\Phi \textit{ is an affine transformation}$$
$$(\textit{translation} + \textit{rotation}).$$

In this case

$$x = L \cdot a + c, \quad c \in \mathbb{R}^3, \quad L \in \mathcal{L}_0(\mathbb{R}^3), \quad L^{-1} = L^T,$$

and

$$u(a) = (L - I)a + c,$$

where $\mathcal{L}_0(\mathbb{R}^3)$ is the space of orthogonal matrices on $\mathbb{R}^3$.

b) Linear compression or elongation

A typical example of elongation is given by the linear stretching of an elastic rod or of a linear spring.

Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$. The uniform elongation in the direction $e = e_1$ reads

$$x_1 = \lambda a_1, \quad x_2 = a_2, \quad x_3 = a_3,$$

with $\lambda > 1$; $0 < \lambda < 1$ would correspond to the uniform compression of a linear spring or an elastic rod. The displacement is then given by $u(a) = [(\lambda - 1)a_1, 0, 0]$ and

$$\nabla\Phi = \begin{pmatrix} \lambda - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + I.$$

c) Shear

We consider here simple shear in two orthogonal directions. Such a deformation occurs, for instance, when one tears a sheet of paper.

The shear in the direction e_1 parallel to the direction e_2 reads

$$\begin{cases} x_1 = a_1 + \rho a_2, \\ x_2 = a_2 + \rho a_1, \\ x_3 = a_3, \end{cases}$$

where $\rho > 0$; hence, the displacement is given by

$$u(a) = \begin{pmatrix} \rho a_2 \\ \rho a_1 \\ 0 \end{pmatrix},$$

and

$$\nabla \Phi = \begin{pmatrix} 0 & \rho & 0 \\ \rho & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + I.$$

Remark 1.1: We will see in what follows that, in some sense, a general deformation can be decomposed into proper elementary deformations of the types above.

1.2. Motion and its observation (kinematics)

Kinematics is the study of the motion of a system related to an observer, which is called the reference system.

With kinematics, we need to introduce two new elements:

- A privileged continuous parameter t corresponding to time. This implies the choice of a chronology; that is, a way to measure time.[1]
- A given system of linear coordinates, or frame of reference, that is "fixed" with respect to the observer. It is defined, in the affine space, by its origin O and three orthonormal basis vectors e_1, e_2, e_3.

[1] From a strictly mathematical point of view, it would not be absurd, for example, to replace t by t^3, but this would change the notion of time interval, and the time $t = 0$ would play a particular role, which it does not.

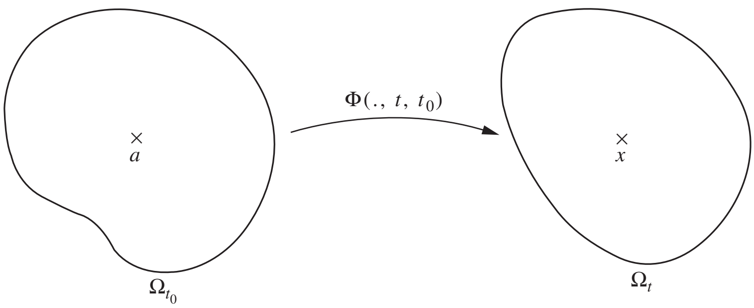

Figure 1.2 The motion of a system.

Definition 1.2. *A reference system is defined by the choice of a chronology and a frame of reference.*[2]

The chronology is fixed once and for all, but, hereafter, we will consider several frames of reference, depending on our objectives.

The motion of the system under consideration is observed during a time interval $I \subset \mathbb{R}$. At each instant $t \in I$, the system fills a domain $\Omega_t \subset \mathbb{R}^3$. The motion is then geometrically defined by a family of deformation mappings, depending on the time $t \in I$ (see Figure 1.2). We denote by $\Phi(t, t_0)$ the diffeomorphism

$$a \in \Omega_{t_0} \mapsto x = \Phi(a, t, t_0) \in \Omega_t,$$

which maps the position a at time t_0 to the position x at time t, and we make the following natural hypotheses:

- $\Phi(t_0, t_0) = I$,
- $\Phi(t', t) \circ \Phi(t, t_0) = \Phi(t', t_0)$, and
- the maps $(t, a) \mapsto \Phi(a, t, t_0)$ are at least of class $\mathcal{C}^1$ (except in the case of shock waves).

Definition 1.3. *A material system is a rigid body if and only if the maps* $\Phi(t, t')$ *are isometries for every* t *and* t'.

Explicit representation of the motion

We are given a reference time t_0, and we choose for point O the origin of the frame of reference in the affine space. The position of the material point M

[2] We will not emphasize any more these very profound considerations, which can lead, depending on the point of view that is pursued, to nonclassic mechanics (e.g., relativity or quantum mechanics).

is defined at each time t by the vector $\overrightarrow{OM}$; we thus write

$$\overrightarrow{OM} = x = \Phi(a, t, t_0),$$

or more simply, omitting t_0,

$$\overrightarrow{OM} = x = \Phi(a, t).$$

Trajectory of a particle

We consider a particle defined by its position a at time t_0. The trajectory of this particle is the curve $\{\Phi(a, t, t_0)\}_{t \in I}$; I is the interval of time during which the motion is observed.

Velocity of a particle

The velocity of the material point M occupying the position x at time t is the vector

$$\boxed{U = U(x, t) = \frac{\partial \Phi}{\partial t}(a, t, t_0).}$$

Remark 1.2: Of course, we consider the derivatives with respect to the given frame of reference, and the velocity is thus considered with respect to the same system.

Property 1.1. *The vector U is independent of the reference time t_0.*

Proof: Let M be a particle occupying the positions x at time t, a at t_0 and a' at t_0' $(t > t_0 > t_0')$. Thus,

$$\overrightarrow{OM} = x = \Phi(a, t, t_0) = \Phi(a', t, t_0'),$$

and

$$a = \Phi(a', t_0, t_0').$$

Thus,

$$x = \Phi[\Phi(a', t_0, t_0'), t, t_0] = \Phi(a', t, t_0'),$$

where a', t_0, and t_0' are fixed. We set

$$h(t) = \Phi[\Phi(a', t_0, t_0'), t, t_0],$$
$$\ell(t) = \Phi(a', t, t_0').$$

We easily verify that $(dh/dt) = (d\ell/dt)$, which accounts for the result.

Acceleration of a particle

The acceleration of the material particle M occupying the position x at time t is the vector

$$\boxed{\gamma = \gamma(x, t) = \frac{\partial^2 \Phi}{\partial t^2}(a, t, t_0).}$$

Property 1.2. *The vector γ is independent of the reference time t_0 (same proof as for the velocity).*

Remark 1.3: Of course, the vectors U and γ depend on the frame of reference and on the chronology we have chosen.

We will see in Section 1.3 why we prefer to write $U(x, t)$ and $\gamma(x, t)$ instead of $U(a, t)$ and $\gamma(a, t)$.

Stream lines

The stream lines are defined at a time t; they are the lines whose tangent at each point is parallel to the velocity vector at this point.

If $U(x, t)$ is the velocity vector at time t and at $x \in \Omega_t$, computing the stream lines is equivalent to solving the differential system

$$\frac{dx_1}{U_1(x_1, x_2, x_3, t)} = \frac{dx_2}{U_2(x_1, x_2, x_3, t)} = \frac{dx_3}{U_3(x_1, x_2, x_3, t)}.$$

In practice, this system can be solved explicitly by analytic methods only rarely. However, its numerical solution on a computer is easy. For computer simulations of flows, these equations are numerically integrated to obtain the stream lines repeatedly in order to visualize the flow (e.g., movie-type animation).

Remark 1.4: The stream lines are different from the trajectories. However, in the case of a stationary motion (defined below) the stream lines and the trajectories coincide.

1.3. Description of the motion of a system: Eulerian and Lagrangian derivatives

Lagrangian description of the motion of a system (description by the trajectories)

The Lagrangian description of the motion is the one we have considered until now. It consists of giving the trajectory of each particle starting from the initial position, for instance from time $t = 0$:

$$x = \Phi(a, t) = \Phi(a, t, 0), \quad a \in \Omega_0.$$

The velocity and acceleration fields are then, respectively, the vector fields $(\partial \Phi/\partial t)(a, t)$ and $(\partial^2 \Phi/\partial t^2)(a, t)$.

This description is too rich for most practical purposes, and it is not used in general. It is, however, very useful for mathematical analysis and for some very specific situations. In general, we prefer to use the Eulerian description of the motion.

Eulerian description of the motion of a system (description by the velocity field)

In this description, we are given at each time t the velocity field $U = U(x, t)$, where $U(x, t)$ is the velocity of the material particle occupying the position x at time t.

Theoretically, we can recover from this velocity field the trajectories and the Lagrangian description of the motion by solving the following differential equations:

$$\begin{cases} \dfrac{dx}{dt} = U(x, t), \\ x(0) = a, \end{cases}$$

the solution of which is $x = x_a(t) = \Phi(a, t)$. Similarly, we can compute the stream lines by solving for x (t being a fixed parameter) the differential system

$$\frac{dx_1}{U_1} = \frac{dx_2}{U_2} = \frac{dx_3}{U_3}.$$

Definition 1.4. *A motion is steady or stationary if and only if $\Omega_t = \Omega_0$, $\forall t$, and the Eulerian velocity field is independent of t, that is, $U(x, t) \equiv U(x)$, $\forall t$.*

We emphasize here the fact that a body undergoing a stationary motion *is not at rest.*

As we said before, the trajectories and streamlines of a stationary flow are the same.

Eulerian and Lagrangian derivatives

Consider a particle M occupying the position a at time $t = 0$ and the position x at time t. A function $f = f(M, t)$ associated with a particle M (e.g., its velocity, acceleration, or other physical quantities to be defined subsequently) can be represented in two different ways during the motion:

$$f(M, t) = g(a, t) \quad \text{or} \quad f(M, t) = h(x, t),$$

where $x = \Phi(a, t)$; that is to say

$$h(\Phi(a, t), t) = g(a, t).$$

Definition 1.5.

1. *The Eulerian derivatives of f are the quantities*
$$\frac{\partial h}{\partial x_i}(x, t), \qquad \frac{\partial h}{\partial t}(x, t).$$
2. *The Lagrangian derivatives of f are the quantities*
$$\frac{\partial g}{\partial a_i}(a, t), \qquad \frac{\partial g}{\partial t}(a, t).$$

In mechanical engineering the derivative $(\partial g/\partial t)(a, t)$ is denoted dh/dt or Dh/Dt (or df/dt or Df/Dt) and is sometimes called the total derivative of f.

We deduce from the relation

$$h(\Phi(a, t), t) = g(a, t),$$

that

$$\frac{\partial g}{\partial a_j}(a, t) = \frac{\partial h}{\partial x_k}\frac{\partial x_k}{\partial a_j} = \frac{\partial h}{\partial x_k}(x, t)\frac{\partial \Phi_k}{\partial a_j}(a, t),$$

where we have used the Einstein summation convention on the indices.

Property 1.3. *We have the following relation:*

$$\boxed{\frac{\partial g}{\partial t} = \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + (U \cdot \nabla)h.}$$