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LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES. 328

Fundamentals of Hyperbolic  
Geometry:  
Selected Expositions

Edited by

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## Contents

Preface	ix
Preface 2005	xi
<b>PART I: NOTES ON NOTES OF THURSTON</b>	<b>1</b>
<i>R.D. Canary, D.B.A. Epstein, P.L. Green</i>	
A New Foreword	3
Chapter I.1. $(G, X)$ -structures	31
I.1.1. $(G, X)$ -structures on a manifold	31
I.1.2. Developing map and holonomy	32
I.1.3. Convexity	34
I.1.4. The developing map and convexity	37
I.1.5. The deformation space	38
I.1.6. Thickenings	41
I.1.7. Varying the structure	44
Chapter I.2. Hyperbolic structures	49
I.2.1. Möbius groups	49
I.2.2. The thick–thin decomposition	50
I.2.3. The nearest point retraction	51
I.2.4. Neighbourhoods of convex hyperbolic manifolds	52
I.2.5. Convex thickenings	56
Chapter I.3. Spaces of hyperbolic manifolds	59
I.3.1. The geometric topology	59
I.3.2. $\varepsilon$ -relations and approximate isometries	66

Chapter I.4. Laminations	76
I.4.1. Geodesic laminations	76
I.4.2. Minimal laminations	80
Chapter I.5. Pleated surfaces	89
I.5.1. Introduction	89
I.5.2. Compactness properties of pleated surfaces	91
I.5.3. Realizations	105
PART II: CONVEX HULLS IN HYPERBOLIC SPACE, A THEOREM OF SULLIVAN, AND MEASURED PLEATED SURFACES	117
<i>D.B.A. Epstein, A. Marden</i>	
Chapter II.1. Convex hulls	121
II.1.1. Introduction	121
II.1.2. Hyperbolic convex hulls	121
II.1.3. The nearest point retraction	122
II.1.4. Properties of hyperbolic convex hulls	125
II.1.5. Metric on convex hull boundary	128
II.1.6. Hyperbolic convex hulls	132
II.1.7. Limits of lines and planes	134
II.1.8. Ridge lines	135
II.1.9. The roof	136
II.1.10. Lowering the roof	138
II.1.11. The bending measure	141
II.1.12. The boundary is a complete hyperbolic manifold	144
II.1.13. Finite approximations to the convex hull boundary	148
II.1.14. Convergence of laminations	150
Chapter II.2. Foliations and the epsilon distant surface	153
II.2.1. Introduction	153
II.2.2. The epsilon distant surface	153
II.2.3. From infinity to the epsilon surface	159
II.2.4. Extending a lamination to a pair of orthogonal foliations	166
II.2.5. Some standard vector fields	170
II.2.6. Lipschitz line fields in the hyperbolic plane	172
II.2.7. Foliation coordinates	176
II.2.8. Formulas in the hyperbolic plane	182

*Contents*

vii

II.2.9. Flat equidistant surfaces	186
II.2.10. Foliations on the convex hull	187
II.2.11. Three orthogonal fields	190
II.2.12. The equidistant surface from a finitely bent convex hull boundary	193
II.2.13. Surfaces equidistant from a general convex hull boundary	195
II.2.14. The map $\rho$	196
II.2.15. Numerical results	208
II.2.16. Counterexample	208
Chapter II.3. Measured pleated surfaces	211
II.3.1. Introduction	211
II.3.2. Finite quakebends	211
II.3.3. Norms	213
II.3.4. Products of rotations about geodesics	214
II.3.5. The quakebend cocycle	217
II.3.6. The quakebend map	219
II.3.7. Invariance	219
II.3.8. Deformations	221
II.3.9. Derivatives	222
II.3.10. Second variation	226
II.3.11. Varying the lamination	228
Appendix	239
Addendum 2005	255
PART III: EARTHQUAKES IN 2-DIMENSIONAL HYPERBOLIC GEOMETRY	267
<i>W.P. Thurston</i>	
Chapter III.1. Earthquakes in 2-dimensional hyperbolic geometry	269
III.1.1. Introduction	269
III.1.2. What are hyperbolic earthquakes?	271
III.1.3. Associating earthquakes to maps of the circle	276
III.1.4. Examples	281
III.1.5. Earthquakes on hyperbolic surfaces	283
III.1.6. The measure and cause of earthquakes	285
III.1.7. Quasi-symmetries and quasi-isometries	287

PART IV: LECTURES ON MEASURES ON LIMIT SETS OF KLEINIAN GROUPS	291
<i>S.J. Patterson</i>	
Chapter IV.1. The problems with which we shall be concerned	293
Chapter IV.2. A measure on the limit set	300
Chapter IV.3. First fruits	308
Chapter IV.4. Spectral theory	314
Chapter IV.5. Geodesic flows	321

## Preface

During the academic year 1983/84, the Science and Engineering Research Council of the United Kingdom gave generous financial support for two symposia, at the Universities of Warwick and Durham, on hyperbolic geometry, Kleinian groups and 3-dimensional topology. The symposium at Durham was also sponsored by the London Mathematical Society. I would like to express my thanks to both the SERC and the LMS for their help and support. It is a pleasure to acknowledge the help of my co-organizer at Durham, Peter Scott, who was also an unofficial co-organizer at Warwick. He made an essential contribution to the great success of the symposia.

The world's foremost contributors to this very active area were all invited, and nearly all of them came. The activity centred on the University of Warwick, and climaxed with a 2-week long intensive meeting at the University of Durham during the first 2-weeks of July 1984. There was earlier a period of intense activity during the Easter vacation of 1984, when a number of short introductory lectures were given. The text of the most important of these series of lectures, by S.J. Patterson, is published in these Proceedings.

The papers published here are the result of an invitation to all those attending the two Symposia to submit papers. Not all the papers submitted were the subject of talks given during the Symposia – the contents of the Proceedings are based on their relevance to the subject, and not on their accuracy as documents recording the events of the Symposia. Also, a number of important contributions to the Symposia are not published here, having been previously promised elsewhere.

One of the few advantages of being an editor is that one can confer certain rights and privileges on oneself. I have taken the opportunity of accepting as suitable for publication several rather large papers of which I was the author or co-author, and which have a substantial element of exposition. This is a field which is expanding very quickly, mainly under Thurston's influence, and more



material of an expository nature is sorely needed, as many of those attempting to penetrate the area will testify. I hope that my own efforts in this direction will be of some help.

Due to the amount of material submitted, it has been necessary to publish two separate books. The division was done on the basis of first sorting into five different fairly narrowly defined subject areas, and then trying to balance the sizes of the two books. The books are entitled “Analytical and Geometric Aspects of Hyperbolic Space” and “Low-Dimensional Topology and Kleinian Groups”.

At an early stage I made the decision to set the books by computer, with the advantage that the entire process would be under my own control. This has been an interesting experience. I will content myself with the comment that computer typesetting is not the joy and wonder that I once thought it would be; the considerable delay in publication has been largely due to the unforeseen difficulties encountered in this process.

My particular thanks go to Russell Quin, without whose help the typesetting difficulties would never have been overcome. I am grateful to Kay Dekker, who has spent many hours creating fonts of special characters needed for this work. I would like to thank the University of Warwick Computer Unit for the use of their facilities for the printing.

Finally I must thank the contributors for their patience and forbearing during the long delay before publication.

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12 July 1986

## Preface 2005

In 1987 Cambridge University Press published the proceedings of meetings in Warwick and Durham as Lecture Notes 111 and 112. The original preface is reprinted above. Taking account of demand, CUP suggested that certain of the articles in the original two volumes be reprinted. We have chosen four of them to comprise a single new volume. We believe they will continue to be helpful to those learning about the field. The authors have kindly given permission to reissue their work.

Part I, “Notes on Notes . . .” (N&N), has a new foreword by Canary. Particular topics in Chapters 8 and 9 of Thurston’s original lecture notes had formed the basis for much of N&N. Still, there is other important material in those chapters that has not been widely digested by the mathematical community at large, yet has turned out to play key roles in later developments. The purpose of the foreword is to provide a guide to the recent literature where explanations of this additional material are now available. Also included are brief accounts of the recently announced solutions of the tameness and ending lamination conjectures, direct generalizations of topics in those fateful chapters.

Likewise, in Part II, the centrality of the convex core in studying hyperbolic manifolds has become even more apparent in the intervening years as important additional details of its structure have been worked out. Accordingly, the careful account of this topic given in the original article on convex hulls and Sullivan’s theorem has been brought up to date with a new addendum by Epstein and Marden.

Part III in this volume is Thurston’s famous paper that develops the notion of “earthquake” and proves the Earthquake Theorem. Earthquake maps are to hyperbolic geometry much like quasi-conformal maps are to complex analysis.

Part IV consists of five lectures by S.J. Patterson that develop from basic principles the theory of measures on limit sets of Kleinian groups. The theory developed applies to discrete groups acting on any dimension hyperbolic space. In particular the exponent of convergence is discussed in reference to the geodesic flow and to the Hausdorff dimension of the limit set. The Sullivan–Patterson measures play an increasing role, in differential geometry, in parts of ergodic theory and in geometric group theory.

The mistakes (that we are aware of) have been corrected. We are grateful to Curt McMullen and to Caroline Series for bringing to our attention a number of infelicities.

Looking back over the 18 years since the original volumes were written, it is amazing how much has been accomplished in the subject. This period could well serve as a model example of collective efforts of the many researchers in a field resulting in a very dramatic increase in the knowledge and depth of understanding of it.

We thank David Tranah for suggesting the reprinting, and for his continuing patient support and encouragement along the way. It was a great relief when he was able to arrange for retyping the original articles into LaTeX. It turns out that automated conversion from Troff to LaTeX only works well with relatively simple files.

Richard D. Canary  
David B.A. Epstein  
Albert Marden

February 2005