Cambridge University Press
978-0-521-61558-7 — Fundamentals of Hyperbolic Manifolds
Edited by R. D. Canary , A. Marden , D. B. A. Epstein
Excerpt
More Information

PART I

Notes on Notes of Thurston

R.D. Canary, Department of Mathematics, University of Michigan,
Ann Arbor, MI 48109, USA

D.B.A. Epstein, Mathematics Institute, University of Warwick,
Conventry CV4 7AL, UK

P.L. Green, Barton Peveril College, Eastleigh, Hampshire SO5 5LX, UK

A New Foreword

*Richard D. Canary**

The article “Notes on Notes of Thurston” was intended as an exposition of some portions of Thurston’s lecture notes *The Geometry and Topology of Three-Manifolds*. The work described in Thurston’s lecture notes revolutionized the study of Kleinian groups and hyperbolic manifolds, and formed the foundation for parts of Thurston’s proof of his Geometrization theorem. At the time, much of the material in those Notes was unavailable in a published form. In this foreword, we point the reader to some more recent publications where detailed explanations of the material in Thurston’s original lecture notes are available. We will place a special emphasis on Thurston’s Chapters 8 and 9. This material was the basis for much of our original article and it still represents the part least well-digested by the mathematical community. This is also the material which has been closest to the author’s subsequent interests, so the selection will, by necessity, reflect some of his personal biases.

We hope this foreword will be useful to students and working mathematicians who are attempting to come to grips with the very beautiful, but also sparingly described, mathematics in Thurston’s notes. No attempt has been made to make this foreword self-contained. It is simply a rough-and-ready guide to some of the relevant literature. In particular, we will not have space to define all the mathematical terms used, but we hope the reader will make use of the many references to sort these out. In particular, we will assume that the reader has a copy of Thurston’s notes on hand. We would also like to suggest that it would be valuable for a publisher to make available Thurston’s lecture notes, in their original form. The author would like to apologize, in advance, to the mathematicians whose relevant articles have been omitted due to the author’s ignorance.

* partially supported by grants from the National Science Foundation

In a final section, we describe some recent progress on the issues dealt with in Chapters 8 and 9 of Thurston's notes.

We would like to thank Francis Bonahon, Al Marden and Yair Minsky for their helpful comments on earlier versions of this foreword.

1. General references

Before focusing in a more detailed manner on the material in Chapters 8 and 9, we will discuss some of the more general references which have appeared since the publication of our article. In order to conserve space, we will be especially telegraphic in this section.

Thurston [153] recently published volume I of a new version of his lecture notes under the title *Three-dimensional Geometry and Topology.* This new volume contains much of the material in Chapter 1, 2, and 3 of the original book, as well as material which comes from Sections 5.3 and 5.10. However, the most exciting and novel portions of his original notes have been left for future volumes. A number of other books on Kleinian groups and hyperbolic manifolds have been published in the last 15 years, including books by Apanasov [14], Benedetti and Petronio [23], Buser [50], Kapovich [83], Katok [84], Maskit [104], Matsuzaki and Taniguchi [105], Ohshika [128] and Ratcliffe [137].

There are now several complete published proofs of Thurston's Geometrization Theorem for Haken 3-manifolds available. McMullen [107] used his proof of Kra's Theta Conjecture to outline a proof of the Geometrization Theorem for Haken 3-manifolds that do not fiber over the circle. A more complete version of this approach is given by Otal [132], who also incorporates work of Barrett and Diller [15]. Kapovich [83] has recently published a book on the proof of the Geometrization Theorem. His approach to the main portion of the proof is based on work of Rips (see [21]) on the actions of groups on **R**-trees. (Morgan and Shalen [120, 121, 122] first used the theory of **R**-trees to prove key portions of the Geometrization Theorem. See Bestvina [20] or Paulin [133] for a more geometric viewpoint on how actions of groups on **R**-trees arise as limits of divergent sequences of discrete faithful representations.) An outline of Thurston's original proof of the main portion of the Geometrization Theorem was given by Morgan in [119]. Portions of this proof are available in Thurston's article [152] and preprint [155]. Thurston's original proof develops much more structural theory of Kleinian groups than the later proofs.

Otal [131] also published a proof of the Geometrization Theorem in the case where the 3-manifold fibers over the circle. Otal's proof makes use of the theory of **R**-trees, and in particular uses a deep theorem of Skora [144] which characterizes certain types of actions of surface groups on **R**-trees. (Kleineidam and Souto [90] used some of Otal's techniques to prove a spectacular generalization of Thurston's Double Limit Theorem to the setting of hyperbolic structures on compression bodies.) Thurston's original proof of the Geometrization Theorem for 3-manifolds which fibre over the circle is available at [154]. (A survey of this proof is given by Sullivan [146]; see also McMullen [108].)

We will now briefly indicate where one might look for details on some of the material in Thurston's notes which is not in Chapters 8 and 9. The material in Sections 4.1–4.7 of Thurston's notes is discussed in Chapter E of Benedetti–Petronio [23]. The material in Sections 4.8 and 4.9 was further developed by Epstein in [65]. The results in Sections 4.10 and 4.11 were generalized in Floyd–Hatcher [73] and Hatcher–Thurston [76].

The material in Section 5.1 is the subject matter of Sections 1.5–1.7 of [57]. In Sections 5.2, 5.5 and 5.6, Thurston develops a useful estimate for the dimension of the representation variety, which was proven carefully by Culler and Shalen in Section 3 of [62]. Thurston's Hyperbolic Dehn Surgery theorem is established in Section 5.8, using the dimension count established in the previous sections and the theory developed in Section 5.1. This version of the proof is discussed in Hodgson–Kerckhoff [77] and, in more detail, in Bromberg [46]. Bromberg also develops generalizations of Thurston's Hyperbolic Dehn Surgery theorem to the infinite volume setting, see also Bonahon–Otal [31] and Comar [61]. A complete proof of the Hyperbolic Dehn Surgery theorem using ideal triangulations is given by Petronio and Porti [135]. The proof of the Mostow–Prasad rigidity theorem given in Section 5.9 follows the same outline as Mostow's original proof [123], see also Marden [98], Mostow [124] and Prasad [136]. In Sections 5.11 and 5.12, Thurston proves Jørgensen's theorem that given a bound C, there exists a finite collection of manifolds, such that every hyperbolic 3-manifold of volume at most C is obtained from one of the manifolds in the collection by Dehn Filling; see also Chapter E in [23].

In Sections 6.1–6.5, Thurston gives Gromov's proof of the Mostow–Prasad rigidity theorem and develops Gromov's theory of simplicial volume; see Gromov [74] and Chapter C of Benedetti and Petronio [23]. In Section 6.6, Thurston proves that the set of volumes of hyperbolic 3-manifolds is well-ordered, again see Chapter E of Benedetti and Petronio [23]. Dunbar and Meyerhoff [64] generalized Thurston's arguments to show that the set of volumes of hyperbolic 3-orbifolds is well-ordered.

Chapter 7 of the original notes, concerning volumes of hyperbolic manifolds, was written by John Milnor and much of the work in this chapter appears in appendices to [110] and [111]. Portions of the material in the incomplete Chapter 11 appear in Appendix B of McMullen [108]. Chapter 13 begins with the theory of orbifolds, see for example Scott [142] and Kapovich [83]. Scott [142] also discusses the orbifold viewpoint on Seifert fibered spaces and the geometrization of Seifert fibered spaces. The remainder of Chapter 13 concerns Andreev's theorem and its generalizations. Andreev's original work appeared in [11] and [12]. Andreev's theorem has been generalized by Rivin–Hodgson [138] and Rivin [139].

2. Chapter 8 of Thurston's notes

Sections 8.1 and 8.2 largely deal with basic properties of the domain of discontinuity and the limit set of a Kleinian group. Variations on this material can be found in any text on Kleinian groups, for example [104] or [105].

2.1. Geometrically finite hyperbolic 3-manifolds

In Section 8.3, Thurston offers a new viewpoint on two of the main results in Marden's seminal paper "The geometry of finitely generated Kleinian groups." Marden's Stability theorem (Proposition 9.1 in [98]) asserts that any small deformation of a convex cocompact Kleinian group is itself convex cocompact and is quasiconformally conjugate to the original group. Thurston's version of this theorem (Proposition 8.3.3 in his notes) appears as Proposition 2.5.1 in our article [57]. Marden's Stability theorem also includes a relative version of this result, which asserts that any small deformation of a geometrically finite Kleinian groups that preserves parabolicity, is itself geometrically finite and is quasiconformally conjugate to the original manifold.

Marden's Isomorphism Theorem (Theorem 8.1 in [98]) asserts that any homotopy equivalence between two geometrically finite hyperbolic 3-manifolds which extends to a homeomorphism of their conformal boundaries, is homotopic to a homeomorphism which lifts (and extends) to a quasiconformal homeomorphism of $\mathbf{H}^3 \cup S^2_\infty$. Thurston's Proposition 8.3.4 is a variation on Marden's Isomorphism theorem.

Proposition 8.3.4: *Let $N_1 = \mathbf{H}^n/\Gamma_1$ and $N_2 = \mathbf{H}^n/\Gamma_2$ be two convex cocompact hyperbolic n-manifolds and let M_1 and M_2 be strictly convex submanifolds of N_1 and N_2. If $\phi\colon M_1 \to M_2$ is a homotopy equivalence which is a homeomorphism from ∂M_1 to ∂M_2, then there exists a map $f\colon\ \mathbf{H}^n \cup S^{n-1}_\infty \to \mathbf{H}^n \cup S^{n-1}_\infty$ such that the restriction $\hat f$ of f to S^{n-1}_∞ is quasiconformal, $\hat f \Gamma_1 \hat f^{-1} = \Gamma_2$, and the restriction of f to $\mathbf{H}^n$ is a quasi-isometry.*

In Section 8.4, Thurston continues his study of geometrically finite hyperbolic 3-manifolds. Theorem 8.4.2 is Ahlfors' result, see [3], that the limit set

$\Lambda(\Gamma)$ of a geometrically finite hyperbolic manifold $N = \mathbf{H}^n/\Gamma$ either has measure zero or is all of the sphere at infinity S^{n-1}_∞ and Γ acts ergodically on S^{n-1}_∞. Ahlfors' Measure Conjecture asserts that this is the case for all finitely generated Kleinian groups. In Section 8.12, Thurston proves Ahlfors' conjecture for freely indecomposable geometrically tame Kleinian groups. Proposition 8.4.3 discusses three equivalent definitions of geometric finiteness. The various definitions of geometric finiteness are treated thoroughly by Bowditch [33].

2.2. Measured laminations and the boundary of the convex core

In Section 8.5, Thurston introduces geodesic laminations and observes that the intrinsic metric on the boundary of the convex core is hyperbolic. This result is established in Chapter 1 of Epstein–Marden [66] and by Rourke [140]. Later, Thurston will observe that the boundary of the convex core is an uncrumpled surface. Uncrumpled surfaces are now known as pleated surfaces. Geodesic laminations are treated in Chapter 4 of our article [57] and in Chapter 4 of Casson-Bleiler [60].

In Section 8.6, Thurston introduces transverse measures on geodesic laminations. In particular, he develops the bending measure on the bending locus of the boundary of the convex core. The bending measure is discussed in Section 1.11 of Epstein–Marden [66]. Measured laminations are discussed by Hatcher [75] and Penner–Harer [134]. The parallel theory of measured foliations is developed in great detail in the book by Fathi, Laudenbach and Poenaru [70]. The connection between measured laminations and measured foliations is made explicit by Levitt [97]. Hubbard and Masur [80] showed that measured foliations can themselves be naturally linked to the theory of quadratic differentials, see also Marden–Strebel [101]. One of the most spectacular applications of the theory of measured laminations was Kerckhoff's proof [87] of the Neilsen Realization Theorem.

Bonahon developed the theory of geodesic currents, which are a generalization of measured laminations, in [25] and [26]. This theory provides a beautiful and flexible conceptual framework for the theory of measured laminations and was put to central use in Bonahon's proof [25] that finitely generated, freely indecomposable Kleinian groups are geometrically tame. Bonahon [26] also used geodesic currents to give a beautiful treatment of Thurston's compactification of Teichmüller space. Bonahon is currently preparing a research monograph [29] which covers geodesic laminations, measured laminations, train tracks and geodesic currents. It also describes Bonahon's more recent work on transverse cocycles and transverse Hölder distributions for geodesic laminations which provide powerful new tools for the study of deformation spaces of hyperbolic manifolds. As one application of these techniques Bonahon [27] has computed the derivative of the function with domain a deformation space of geometrically finite hyperbolic 3-manifolds given by considering the volumes of the convex cores. His formula is a generalization of Schläfli's formula for the variation of volumes of hyperbolic polyhedra. Bonahon's briefer survey paper [28] covers some of the same material; both the research monograph and the survey paper are highly recommended.

2.3. Quasifuchsian groups and bending

In Section 8.7, Thurston begins his study of quasifuchsian groups. A finitely generated, torsion-free Kleinian group is said to *quasifuchsian* if its limit set is a Jordan curve and both components of its domain of discontinuity are invariant under the entire group. Thurston's definition of a quasifuchsian group is incomplete as it leaves out the condition on the domain of discontinuity. His definition allows Kleinian groups which uniformize twisted I-bundles over surfaces, as well as those which uniformize product I-bundles. Proposition 8.7.2 offers several equivalent definitions of quasifuchsian groups. We give a corrected version of Thurston Proposition 8.7.2 below:

Proposition 8.7.2. (Maskit [102]) *If Γ is a finitely generated, torsion-free Kleinian group, then the following conditions are equivalent:*

(1) Γ *is quasifuchsian.*
(2) *The domain of discontinuity $\Omega(\Gamma)$ of Γ has exactly two components, each of which is invariant under the entire group.*
(3) Γ *is quasiconformally conjugate to a Fuchsian group, i.e. there exists a Fuchsian group $\Theta \subset \mathrm{PSL}_2(\mathbf{R})$ (such that its limit set $\Lambda(\Theta) = \mathbf{R} \cup \infty$) and a quasiconformal map $\phi\colon \hat{\mathbf{C}} \to \hat{\mathbf{C}}$ such that $\Gamma = \phi\Theta\phi^{-1}$.*

This characterization is originally due to Maskit, see Theorem 2 in [102], although Thurston follows the alternative proof given by Marden in Section 3 of [98].

Example 8.7.3 is the famous Mickey Mouse example, which is produced using the bending construction. Bending has been studied extensively by Apanasov [14] and Tetenov [13], Johnson and Millson [81], Kourouniotis [91] and others. Universal bounds on the bending lamination of a quasifuchsian group and hence on the bending deformation, are obtained by Bridgeman [34, 35] (and generalized to other settings by Bridgeman–Canary [37]). These bounds are discussed in more detail in the addendum to Epstein–Marden [66] in this volume.

After the Mickey Mouse example, Thurston discusses simplicial hyperbolic surfaces, although he does not give them a name. A simplicial hyperbolic surface is a map, not necessarily an embedding, of a triangulated surface into a 3-manifold such that each face is mapped totally geodesically and the total angle around each vertex is at least 2π. The restriction on the vertices guarantees that the induced metric, usually singular, on the surface has curvature ≤ -1 in the sense of Alexandrov. Simplicial hyperbolic surfaces were used extensively by Bonahon [25] in his proof that freely indecomposable Kleinian groups are geometrically tame and they are discussed in detail in Section 1.3 of [25].

Proposition 8.7.7 asserts that every complete geodesic lamination is realizable in a quasifuchsian hyperbolic 3-manifold. This statement is included in Theorem 5.3.11 in [57]. We will discuss realizability of laminations more fully when we come to Sections 8.10 and 9.7.

2.4. Pleated surfaces and realizability of laminations

Section 8.8 of Thurston's notes concerns pleated surfaces, which are called uncrumpled surfaces in the notes. The results in this section form the basis of Section 5 of our original article [57]. Pleated surfaces are also discussed in Thurston's articles on the Geometrization theorem [152, 154, 155].

Section 8.9 of Thurston's notes develops the theory of train tracks. Proposition 8.9.2 and Corollary 8.9.3 assert that any geodesic lamination on a surface may be well-approximated by a train track. Three-dimensional versions of these results play a key role in Bonahon's work and Section 5 of his paper [25] discusses train track approximations to geodesic laminations in great detail. The general theory of train tracks is developed by Penner and Harer in [134].

In Section 8.10, Thurston turns to the issue of realizability of laminations in 3-manifolds. We discuss this issue in detail in Section 5.3 of [57]. One begins with an incompressible, type-preserving map $f\colon S \to N$ of a finite area hyperbolic surface S into a hyperbolic 3-manifold N. (An incompressible map $f\colon S \to N$ is said to be type-preserving if $f_*(g)$ is parabolic if and only if $g \in \pi_1(S)$ is parabolic where $f_*\colon \pi_1(S) \to \pi_1(N)$ is regarded as a map between the associated groups of covering transformations.) One says that a geodesic lamination λ on S is *realizable* if there is a pleated surface $h\colon S \to N$ which maps λ into N in a totally geodesic manner. If a realization exists then the image of λ is unique (Proposition 8.10.2 in Thurston and Lemma 5.3.5 in [57].) The map from the space of pleated surfaces (which are homotopic to f) into the space of geodesic laminations (with the Thurston topology) given by taking a pleated surface to its pleating locus is continuous (Proposition 8.10.4 in Thurston and Lemma 5.3.2 in [57].) Propositions 8.10.5, 8.10.6 and 8.10.7 develop more basic properties of geodesic measured laminations, see the earlier references for details. Theorem 8.10.8 in Thurston's notes asserts that the set $\mathcal{R}_f$ of realizable laminations is open and dense in the set $GL(S)$ of all geodesic laminations on S, see Theorem 5.3.10 in [57] for details. Thurston's Corollary 8.10.9, which asserts that if N is geometrically finite then $\mathcal{R}_f = GL(S)$ unless N virtually fibres over the circle with fibre $f(S)$, is stated as Corollary 5.3.12 in [57]. We note that Thurston's Conjecture 8.10.10, which asserts that $f_*(\pi_1(S))$ is quasifuchsian if and only if $\mathcal{R}_f = GL(S)$, is a consequence of Bonahon's work [25], see the discussion after Proposition 9.7.1 and the discussion of Bonahon's work in Section 4.

In related work, Brock [38] proved that the length function is continuous on the space of realizable laminations in $AH(S) \times ML(S)$ and extends to a continuous function on all of $AH(S) \times ML(S)$. Thurston claimed this result and used it in his proof [154] of the Geometrization theorem for 3-manifolds which fiber over the circle.

2.5. Relative compact cores and ends of hyperbolic 3-manifolds

It will be convenient to formalize the material in Section 8.11 in the language of relative compact cores. If N is a hyperbolic 3-manifold, and we choose ε less than the Margulis constant (see Section 4.5 of Thurston [153] or Chapter D in Benedetti–Petronio [23] for example) we can define the ε-thin part of N to be the portion of N with injectivity radius at most ε. Each compact part of the ε-thin part will be a solid torus neighborhood of a geodesic, while each non-compact component will be the quotient of a horoball by a group of parabolic isometries (isomorphic to either $\mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$). We obtain N^0 from N by removing its "cusps", i.e. the non-compact components of its thin part. A relative compact core M for N is a compact 3-dimensional submanifold of N^0 whose inclusion into N is a homotopy equivalence which intersects each toroidal component of ∂N^0 in the entire torus and intersects each annular component of ∂N^0 in a single incompressible annulus. Bonahon [24], McCullough [106] and Kulkarni–Shalen [93] proved that every hyperbolic 3-manifold with finitely generated fundamental group admits a relative compact core.

Feighn–McCullough [71] and Kulkarni–Shalen [93] (see also Abikoff [1]) have used the relative compact core to give topological proofs of Bers' area inequality, which asserts that the area of the conformal boundary is bounded by the number of generators (see Bers [16]) and Sullivan's Finiteness Theorem, which asserts that the number of conjugacy classes of maximal parabolic subgroups of a Kleinian group is bounded by the number of generators (see Sullivan [148]). See Section 7 of Marden [98] for a similar treatment of Bers' area inequalities in the setting of geometrically finite groups where Marden constructs an analogue of the relative compact core.

Much of the remainder of Sections 8 and 9 are taken up with understanding the geometry and topology of ends of hyperbolic 3-manifolds. Ends of N^0 are in a one-to-one correspondence with components of $\partial M - \partial N^0$, see Proposition 1.3 in [25], where M is a relative compact core for N. An end of N^0 is *geometrically finite* if it has a neighborhood which does not intersect the convex core. At the end of Section 8.11, Thurston introduces the crucial notion of a simply degenerate end of a hyperbolic 3-manifold. If M is a relative compact core for N, then an end E of N^0 which has a neighborhood bounded by an incompressible component S of $\partial M - \partial N^0$ is said to be *simply degenerate* if there exists a sequence $\{\gamma_i\}$ of non-trivial simple closed curves on S whose geodesic representatives in N all lie in the component of $N^0 - M$ bounded by S and leave every compact subset of N. (Here we have given Bonahon's version of Thurston's definition, which is equivalent to Thurston's.) A hyperbolic 3-manifold in which each component of $\partial M - \partial N^0$ is incompressible is said to be *geometrically tame* if each of its ends is either geometrically finite or simply degenerate.

We will say that the relative compact core M has *relatively incompressible* boundary if each component of $M - \partial N^0$ is incompressible. Thurston works almost entirely in the setting of hyperbolic 3-manifolds whose relative compact core has relatively incompressible boundary. If N has no cusps, the relative compact core has incompressible boundary if and only if $\pi_1(N)$ is freely indecomposable. In general, the relative compact core has relatively incompressible boundary if and only if there does not exist a non-trivial free decomposition of $\pi_1(N)$ such that every parabolic element is conjugate into one of the factors, see Proposition 1.2 in Bonahon or Lemma 5.2.1 in Canary-McCullough [58].

In Section 4.1, we will explain how the definition of geometric tameness is extended to all hyperbolic 3-manifolds with finitely generated fundamental group.

2.6. Analytic consequences of tameness

In Section 8.12, Thurston proves a minimum principle for positive superharmonic functions on geometrically tame hyperbolic 3-manifolds.

Theorem 8.12.3: *If N is a geometrically tame hyperbolic 3-manifold (whose compact core has relatively incompressible boundary), then for every non-constant positive superharmonic (i.e. $\Delta h \leq 0$) function h on N,*

$$\inf_{C(N)} h = \inf_{\partial C(N)} h$$