1 Euclidean geometry

This chapter discusses the geometry of n-dimensional Euclidean space $\mathbb{E}^n$, together with its distance function. The distance gives rise to other notions such as angles and congruent triangles. Choosing a Euclidean coordinate frame, consisting of an origin O and an orthonormal basis of vectors out of O, leads to a description of $\mathbb{E}^n$ by coordinates, that is, to an identification $\mathbb{E}^n = \mathbb{R}^n$.

A map of Euclidean space preserving Euclidean distance is called a *motion* or *rigid body motion*. Motions are fun to study in their own right. My aims are

(1) to describe motions in terms of linear algebra and matrixes;
(2) to find out how many motions there are;
(3) to describe (or classify) each motion individually.

I do this rather completely for $n = 2, 3$ and some of it for all n. For example, the answer to (2) is that all points of $\mathbb{E}^n$, and all sets of orthonormal coordinate frames at a point, are equivalent: given any two frames, there is a unique motion taking one to the other. In other words, any point can serve as the origin, and any set of orthogonal axes as the coordinate frames. This is the geometric and philosophical principle that space is homogeneous and isotropic (the same viewed from every point and in every direction). The answer to (3) in $\mathbb{E}^2$ is that there are four types of motions: translations and rotations, reflections and glides (Theorem 1.14).

The chapter concludes with some elementary sample theorems of plane Euclidean geometry.

1.1 The metric on $\mathbb{R}^n$

Throughout the book, I write $\mathbb{R}^n$ for the vector space of n-tuples $(x_1, \ldots, x_n)$ of real numbers. I start by discussing its metric geometry. The familiar Euclidean distance function on $\mathbb{R}^n$ is defined by

$$|\mathbf{x} - \mathbf{y}| = \sqrt{\left(\sum (x_i - y_i)^2\right)}, \quad \text{where } \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}. \qquad (1)$$

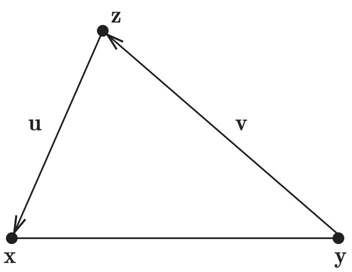

Figure 1.1 Triangle inequality.

The relationship between this distance function and the Euclidean inner product (or dot product) $\mathbf{x} \cdot \mathbf{y} = \sum x_i y_i$ on $\mathbb{R}^n$ is discussed in Appendix B.2. The more important point is that the Euclidean distance (1) is a metric on $\mathbb{R}^n$. If you have not yet met the idea of a *metric* on a set X, see Appendix A; for now recall that it is a distance function $d(\mathbf{x}, \mathbf{y})$ satisfying positivity, symmetry and the triangle inequality. Both the positivity $|\mathbf{x} - \mathbf{y}| \geq 0$ and symmetry $|\mathbf{x} - \mathbf{y}| = |\mathbf{y} - \mathbf{x}|$ are immediate, so the point is to prove the triangle inequality (Figure 1.1).

Theorem (Triangle inequality)

$$|\mathbf{x} - \mathbf{y}| \leq |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}|, \quad \textit{for all } \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n, \tag{2}$$

with equality if and only if $\mathbf{z} = \mathbf{x} + \lambda(\mathbf{y} - \mathbf{x})$ *for* λ *a real number between* 0 *and* 1.

Proof Set $\mathbf{x} - \mathbf{z} = \mathbf{u}$ and $\mathbf{z} - \mathbf{y} = \mathbf{v}$ so that $\mathbf{x} - \mathbf{y} = \mathbf{u} + \mathbf{v}$; then (2) is equivalent to

$$\sqrt{\sum u_i^2} + \sqrt{\sum v_i^2} \geq \sqrt{\left(\sum (u_i + v_i)^2\right)}. \tag{3}$$

Note that both sides are nonnegative, so that squaring, one sees that (3) is equivalent to

$$\begin{aligned} \sum u_i^2 + \sum v_i^2 + 2\sqrt{\left(\sum u_i^2 \cdot \sum v_i^2\right)} &\geq \sum (u_i + v_i)^2 \\ &= \sum u_i^2 + \sum v_i^2 + 2\sum u_i v_i. \end{aligned} \tag{4}$$

Cancelling terms, one sees that (4) is equivalent to

$$\sqrt{\left(\sum u_i^2 \cdot \sum v_i^2\right)} \geq \sum u_i v_i. \tag{5}$$

If the right-hand side is negative then (5), hence also (2), is true and strict. If the right-hand side of (5) is ≥ 0 then it is again permissible to square both sides, giving

$$\sum u_i^2 \cdot \sum v_j^2 \geq \left(\sum u_i v_i\right)\left(\sum u_j v_j\right). \tag{6}$$

You will see at once what is going on if you write this out explicitly for $n = 2$ and expand both sides. For general n, the trick is to use two different dummy indexes i, j as in (6): expanding and cancelling gives that (6) is equivalent to

$$\sum_{i>j}(u_i v_j - u_j v_i)^2 \geq 0. \tag{7}$$

Now (7) is true, so retracing our steps back through the argument gives that (2) is true. Finally, equality in (2) holds if and only if $u_i v_j = u_j v_i$ for all i, j (from (7)) and $\sum u_i v_i \geq 0$ (from the right-hand side of (5)); that is, $\mathbf{u}$ and $\mathbf{v}$ are proportional, $\mathbf{u} = \mu\mathbf{v}$ with $\mu \geq 0$. Rewriting this in terms of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ gives the conclusion. QED

1.2 Lines and collinearity in $\mathbb{R}^n$

There are several ways of defining a line (already in the usual x, y plane $\mathbb{R}^2$); I choose one definition for $\mathbb{R}^n$.

Definition Let $\mathbf{u} \in \mathbb{R}^n$ be a fixed point and $\mathbf{v} \in \mathbb{R}^n$ a nonzero direction vector. The *line* L starting at $\mathbf{u} \in \mathbb{R}^n$ with direction vector $\mathbf{v}$ is the set

$$L := \left\{\mathbf{u} + \lambda\mathbf{v} \;\middle|\; \lambda \in \mathbb{R}\right\} \subset \mathbb{R}^n.$$

Three distinct points $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ are *collinear* if they are on a line.

If I choose the starting point $\mathbf{x}$, and the direction vector $\mathbf{v} = \mathbf{y} - \mathbf{x}$, then $L = \{(1-\lambda)\mathbf{x} + \lambda\mathbf{y}\}$. To say that distinct points $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are collinear means that $\mathbf{z} = \{(1-\lambda)\mathbf{x} + \lambda\mathbf{y}\}$ for some λ. Writing

$$[\mathbf{x}, \mathbf{y}] = \left\{\mathbf{x} + \lambda(\mathbf{y} - \mathbf{x}) \;\middle|\; 0 \leq \lambda \leq 1\right\}$$

for the line segment between $\mathbf{x}$ and $\mathbf{y}$, the possible orderings of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ on the line L are controlled by

$$\left.\begin{array}{r} \lambda \leq 0 \\ 0 \leq \lambda \leq 1 \\ 1 \leq \lambda \end{array}\right\} \iff \left\{\begin{array}{l} \mathbf{x} \in [\mathbf{z}, \mathbf{y}] \\ \mathbf{z} \in [\mathbf{x}, \mathbf{y}] \\ \mathbf{y} \in [\mathbf{x}, \mathbf{z}]. \end{array}\right.$$

Together with the triangle inequality Theorem 1.1, this proves the following result.

Corollary *Three distinct points* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *are collinear if and only if (after a permutation of* $\mathbf{x}, \mathbf{y}, \mathbf{z}$ *if necessary)*

$$|\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}| = |\mathbf{x} - \mathbf{z}|.$$

In other words, collinearity is determined by the metric.

1.3 Euclidean space $\mathbb{E}^n$

After these preparations, I am ready to introduce the main object of study: *Euclidean n-space* $(\mathbb{E}^n, d)$ is a metric space (with metric d) for which there exists a bijective map $\mathbb{E}^n \to \mathbb{R}^n$, such that if $P, Q \in \mathbb{E}^n$ are mapped to $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then

$$d(P, Q) = |\mathbf{y} - \mathbf{x}|.$$

In other words, $(\mathbb{E}^n, d)$ is isometric to the vector space $\mathbb{R}^n$ with its usual distance function, if you like this kind of language.

Since lines and collinearity in $\mathbb{R}^n$ are characterised purely in terms of the Euclidean distance function, these notions carry over to $\mathbb{E}^n$ without any change: three points of $\mathbb{E}^n$ are collinear if they are collinear for some isometry $\mathbb{E}^n \to \mathbb{R}^n$ (hence for all possible isometries); the lines of $\mathbb{E}^n$ are the lines of $\mathbb{R}^n$ under any such identification. For example, for points $P, Q \in \mathbb{E}^n$, the line segment $[P, Q] \subset \mathbb{E}^n$ is the set

$$[P, Q] = \left\{R \in \mathbb{E}^n \;\middle|\; d(P, R) + d(R, Q) = d(P, Q)\right\} \subset \mathbb{E}^n.$$

Remark The main point of the definition of $\mathbb{E}^n$ is that the map $\mathbb{E}^n \to \mathbb{R}^n$ identifying the metrics is not fixed throughout the discussion; I only insist that one such isometry should exist. A particular choice of identification preserving the metric is referred to as a choice of *(Euclidean) coordinates*. Points of $\mathbb{E}^n$ will always be denoted by capital letters P, Q; once I choose a bijection, the points acquire *coordinates* $P = (x_1, \dots, x_n)$. In particular, any coordinate system distinguishes one point of $\mathbb{E}^n$ as the origin $(0, \dots, 0)$; however, different identifications pick out different points of $\mathbb{E}^n$ as their origin. If you want to have a Grand Mosque of Mecca or a Greenwich Observatory, you must either receive it by Divine Grace or make a deliberate extra choice. The idea of space ought to make sense without a coordinate system, but you can always fix one if you like.

You can also look at this process from the opposite point of view. Going from $\mathbb{R}^n$ to $\mathbb{E}^n$, I forget the distinguished origin $0 \in \mathbb{R}^n$, the standard coordinate system, and the vector space structure of $\mathbb{R}^n$, remembering only the distance and properties that can be derived from it.

1.4 Digression: shortest distance

As just shown, the metric of Euclidean space $\mathbb{E}^n$ determines the lines. This section digresses to discuss the idea summarised in the well known cliché ‘a straight line is the shortest distance between two points’; while logically not absolutely essential in this chapter, this idea is important in the philosophy of Euclidean geometry (as well as spherical and hyperbolic geometry).

Principle *The distance $d(P, Q)$ between two points $P, Q \in \mathbb{E}^n$ is the length of the shortest curve joining P and Q. The line segment $[P, Q]$ is the unique shortest curve joining P, Q.*

Sketch proof This looks obvious: if a curve C strays off the straight and narrow to some point $R \notin [P, Q]$, its length is at least

$$d(P, R) + d(R, Q) > d(P, Q).$$

The statement is, however, more subtle: for instance, it clearly does not make sense without a definition of a curve C and its length. A curve C in $\mathbb{E}^n$ from P to Q is a family of points $R_t \in \mathbb{E}^n$, depending on a 'time variable' t such that $R_0 = P$ and $R_1 = Q$. Clearly R_t should at least be a *continuous* function of t – if you allow instantaneous 'teleporting' between far away points, you can obviously get arbitrarily short paths.

The proper definition of curves and lengths of curves belongs to differential geometry or analysis. Given a 'sufficiently smooth' curve, you can define its length as the integral $\int_C \mathrm{d}s$ along C of the infinitesimal arc length $\mathrm{d}s$, given by $\mathrm{d}s^2 = \sum_{i=1}^n \mathrm{d}x_i^2$. Alternatively, you can mark out successive points $P = R_0, R_1, \dots, R_{N+1} = Q$ along the curve, view the sum $\sum_{i=0}^N d(R_i, R_{i+1})$ as an approximation to the length of C, and define the length of C to be the supremum taken over all such piecewise linear approximations. To avoid the analytic details (which are not at all trivial!), I argue under the following weak assumption: under any reasonable definition of the length of C,

for any $\varepsilon > 0$, the curve C can be closely approximated by a piecewise linear path made up of short intervals $[P, R_1]$, $[R_1, R_2]$, etc., such that

$$\text{length of } C \ \geq\ \text{sum of the lengths of the intervals} - \varepsilon.$$

However, by the triangle inequality $d(P, R_2) \leq d(P, R_1) + d(R_1, R_2)$, so that the piecewise linear path can only get shorter if I omit R_1. Dealing likewise with R_2, R_3, etc., it follows that the length of C is $\geq d(P, Q) - \varepsilon$. Since this is true for any $\varepsilon > 0$, it follows that the length of C is $\geq d(P, Q)$. Thus the line interval $[P, Q]$ joining P, Q is the shortest path between them, and its length is $d(P, Q)$ by definition. QED

1.5 Angles

The geometric significance of the Euclidean inner product $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$ on $\mathbb{R}^n$ (Section B.2) is that the inner product measures the size of the *angle* $\angle\mathbf{xyz}$ based at $\mathbf{y}$ for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$:

$$\cos(\angle\mathbf{xyz}) = \frac{(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{z} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}||\mathbf{z} - \mathbf{y}|}. \tag{8}$$

By convention, I usually choose the angle to be between 0 and π. In particular, the vectors $\mathbf{x} - \mathbf{y}$, $\mathbf{z} - \mathbf{y}$ are *orthogonal* if $(\mathbf{x} - \mathbf{y}) \cdot (\mathbf{z} - \mathbf{y}) = 0$.

The notion of angle is easily transported to Euclidean space $\mathbb{E}^n$. Namely, the angle spanned by three points of $\mathbb{E}^n$ is defined to be the corresponding angle in $\mathbb{R}^n$ under a choice of coordinates. The angle is independent of this choice, because the inner product in $\mathbb{R}^n$ is determined by the quadratic form (Proposition B.1), and so ultimately

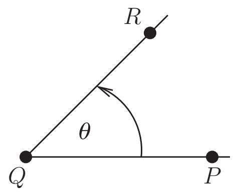

Figure 1.5 Angle with direction.

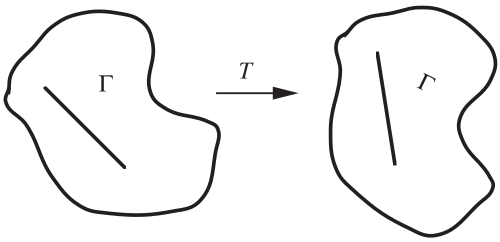

Figure 1.6 Rigid body motion.

by the metric of $\mathbb{E}^n$. In other words, the notion of angle is intrinsic to the geometry of $\mathbb{E}^n$.

There is one final issue to discuss regarding angles that is specific to the Euclidean plane $\mathbb{E}^2$. Namely, *once I fix a specific coordinate system* in $\mathbb{E}^2$, angles $\angle PQR$ acquire a *direction* as well as a size, once we agree (as we usually do) that an anticlockwise angle counts as positive, and a clockwise angle as negative. In Figure 1.5,

$$\angle PQR = -\angle RQP = \theta.$$

Under this convention, angles lie between $-\pi$ and π. Of course formula (8) does not reveal the sign as $\cos\theta = \cos(-\theta)$. It is important to realise that the direction of the angle is not intrinsic to $\mathbb{E}^2$, since a different choice of coordinates may reverse the sign.

1.6 Motions

A *motion* $T : \mathbb{E}^n \to \mathbb{E}^n$ is a transformation that preserves distances; that is, T is bijective, and

$$d(T(P), T(Q)) = d(P, Q) \quad \text{for all } P, Q \in \mathbb{E}^n.$$

The word motion is short for *rigid body motion*; it is alternatively called *isometry* or *congruence*. To say that T preserves distances means that there is 'no squashing or bending', hence the term rigid body motion; see Figure 1.6.

I study motions in terms of coordinates. After a choice of coordinates $\mathbb{E}^n \to \mathbb{R}^n$, a motion T gives rise to a map $T : \mathbb{R}^n \to \mathbb{R}^n$, its coordinate expression, which satisfies

$$|T(\mathbf{x}) - T(\mathbf{y})| = |\mathbf{x} - \mathbf{y}| \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

The first thing I set out to do is to get from the abstract 'preserves distance' definition of a motion to the concrete coordinate expression $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ with A an orthogonal matrix. In the case of the Euclidean plane $\mathbb{E}^2$, the result is even more concrete; A is either a rotation matrix or a reflection matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

1.7 Motions and collinearity

Proposition *A motion $T : \mathbb{E}^n \to \mathbb{E}^n$ preserves collinearity of points, so it takes lines to lines.*

Proof $P, Q, R \in E^n$ are collinear if and only if, possibly after a permutation of P, Q, R,

$$d(P, R) + d(R, Q) = d(P, Q).$$

But T preserves the distance function, so this happens if and only if, possibly after a permutation,

$$d(T(P), T(R)) + d(T(R), T(Q)) = d(T(P), T(Q))$$

which is equivalent to $T(P), T(Q), T(R)$ collinear. QED

The point is of course that, as we saw in 1.3, collinearity can be defined purely in terms of distance; since a motion T preserves distance, it preserves collinearity.

1.8 A motion is affine linear on lines

Proposition *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a motion expressed in coordinates, then*

$$T((1-\lambda)\mathbf{x} + \lambda\mathbf{y}) = (1-\lambda)T(\mathbf{x}) + \lambda T(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$.

Proof A calculation based on the same idea as the previous proof: let $\mathbf{z} = (1-\lambda)\mathbf{x} + \lambda\mathbf{y}$. If $\mathbf{x} = \mathbf{y}$ there is nothing to prove; set $d = |\mathbf{x} - \mathbf{y}|$. Assume first that $\lambda \in [0, 1]$, so that $\mathbf{z} \in [\mathbf{x}, \mathbf{y}]$. Then, as in the previous proposition, $T(\mathbf{z}) \in [T(\mathbf{x}), T(\mathbf{y})]$, so $T(\mathbf{z}) = (1-\mu)T(\mathbf{x}) + \mu T(\mathbf{y})$ for some μ. But $|\mathbf{z} - \mathbf{x}| = \lambda d$, so $T(\mathbf{z})$ is the point at distance $(1-\lambda)d$ from $T(\mathbf{y})$ and λd from $T(\mathbf{x})$, that is, $\mu = \lambda$.

If $\lambda < 0$, say, then $\mathbf{x} \in [\mathbf{y}, \mathbf{z}]$ with $\mathbf{x} = (1-\lambda')\mathbf{y} + \lambda'\mathbf{z}$ and the same argument gives $T(\mathbf{x}) = (1-\lambda')T(\mathbf{y}) + \lambda' T(\mathbf{z})$, and you can derive the statement as an easy exercise. (The point is to write λ' as a function of λ; see Exercise 1.3.) QED

1.9 Motions are affine transformations

Definition A map $T : \mathbb{E}^n \to \mathbb{E}^n$ is an *affine transformation* if it is given in a coordinate system by $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, where $A = (a_{ij})$ is an $n \times n$ matrix with nonzero determinant and $\mathbf{b} = (b_i)$ a vector; in more detail,

$$\mathbf{x} = (x_i) \mapsto \mathbf{y} = \Big(\sum_{j=1}^{n} a_{ij}x_j + b_i\Big), \quad \text{or} \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \tag{9}$$

Proposition *Let $T : \mathbb{E}^n \to \mathbb{E}^n$ be any map. Equivalent conditions:*

(1) *T is given in some coordinate system by $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for A an $n \times n$ matrix.*
(2) *For all vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $\lambda, \mu \in \mathbb{R}$ we have*

$$T\big(\lambda\mathbf{x} + \mu\mathbf{y}\big) - T(0) = \lambda\big(T(\mathbf{x}) - T(0)\big) + \mu\big(T(\mathbf{y}) - T(0)\big).$$

(3) *For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$*

$$T\big((1-\lambda)\mathbf{x} + \lambda\mathbf{y}\big) = (1-\lambda)T(\mathbf{x}) + \lambda T(\mathbf{y}).$$

that is, T is affine linear when restricted to any line.

Discussion The point of the proposition is that condition (3) is a priori much weaker than the other two; it only requires that the map T is affine when restricted to lines. Note also that using the origin 0 in (2) seems to go against my expressed wisdom that there is no distinguished origin in the geometry of $\mathbb{E}^n$. However, recall that any point $P \in \mathbb{E}^n$ can serve as origin after a suitable translation.

Proof (1) $\implies$ (2) is an easy exercise. (2) means exactly that if after performing T we translate by minus the vector $\mathbf{b} = T(0)$ to take $T(0)$ back to 0, then T becomes a linear map of vector spaces. Thus (2) $\implies$ (1) comes from the standard result of linear algebra expressing a linear map as a matrix.

(3) is just the particular case $\lambda + \mu = 1$ of (2). Thus the point of the proposition is to prove (3) $\implies$ (2).

Statement (2) concerns only the 2-dimensional vector subspace spanned by $\mathbf{x}, \mathbf{y} \in V$. We use statement (3) on the two lines $0\mathbf{x}$ and $0\mathbf{y}$ (see Figure 1.9), to get

$$T(2\lambda\mathbf{x}) = (1 - 2\lambda)T(0) + 2\lambda T(\mathbf{x})$$

and

$$T(2\mu\mathbf{y}) = (1 - 2\mu)T(0) + 2\mu T(\mathbf{y}).$$

Now apply (3) again to the line spanned by $2\lambda\mathbf{x}$ and $2\mu\mathbf{y}$:

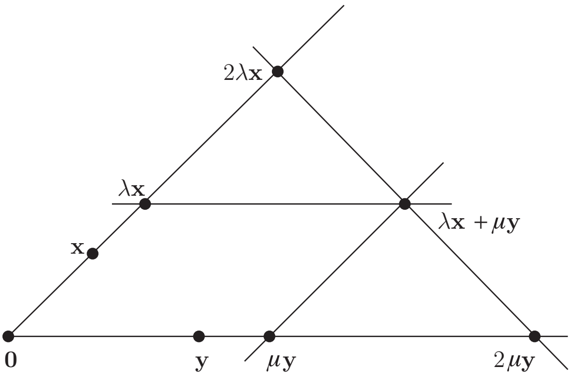

Figure 1.9 Affine linear construction of $\lambda\mathbf{x}+\mu\mathbf{y}$.

$$\begin{aligned} T\big(\lambda\mathbf{x}+\mu\mathbf{y}\big) &= \frac{1}{2}T(2\lambda\mathbf{x})+\frac{1}{2}T(2\mu\mathbf{y}) \\ &= \frac{1}{2}\big((1-2\lambda)T(0)+2\lambda T(\mathbf{x})\big)+\frac{1}{2}\big((1-2\mu)T(0)+2\mu T(\mathbf{y})\big) \\ &= T(0)+\lambda\big(T(\mathbf{x})-T(0)\big)+\mu\big(T(\mathbf{y})-T(0)\big), \end{aligned}$$

as required. QED

Remark Dividing by 2 here is just for the sake of an easy life: $\{\frac{1}{2},\frac{1}{2}\}$ is a convenient solution of $\lambda+\mu=1$. The point is just that $\lambda\mathbf{x}+\mu\mathbf{y}$ lies on a line containing chosen points of $0\mathbf{x}$ and $0\mathbf{y}$. The argument for (3) $\implies$ (2) can be made to work provided every line has ≥ 3 points, that is, over any field with >2 elements.

Corollary *A Euclidean motion $T:\mathbb{E}^n\to\mathbb{E}^n$ is an affine transformation, given in any choice of coordinates $\mathbb{E}^n\to\mathbb{R}^n$ by $T(\mathbf{x})=A\mathbf{x}+\mathbf{b}$.*

This follows at once from Proposition 1.7, the implication (3) $\implies$ (1) in the previous proposition, and the fact that T is bijective, so the matrix A must be invertible.

1.10 Euclidean motions and orthogonal transformations

This section makes a brief use of the relationship between the standard quadratic form $|\mathbf{x}|^2=\sum x_i^2$ on $\mathbb{R}^n$ and the associated inner product $\mathbf{x}\cdot\mathbf{y}=\sum x_i y_i$. If this is not familiar to you, I refer you once again to Appendix B for a general discussion.

Proposition *Let A be an $n\times n$ matrix and $T:\mathbb{R}^n\to\mathbb{R}^n$ the map defined by $\mathbf{x}\mapsto A\mathbf{x}$. Then the following are equivalent conditions:*

(1) *T is a motion $T:\mathbb{E}^n\to\mathbb{E}^n$.*
(2) *A preserves the quadratic form; that is, $|A\mathbf{x}|=|\mathbf{x}|$ for all $\mathbf{x}\in\mathbb{R}^n$.*
(3) *A is an* orthogonal matrix*; that is, it satisfies ${}^{\mathrm{t}}AA=I_n$.*

Proof (1) $\implies$ (2) is trivial. Conversely,

$$|A\mathbf{x} - A\mathbf{y}|^2 = |A(\mathbf{x} - \mathbf{y})|^2 = |\mathbf{x} - \mathbf{y}|^2,$$

where the first equality is linearity, and the second follows from (2). Thus T preserves length, so it is a motion. (2) $\iff$ (3) is proved in Proposition B.4, where you can also read more about orthogonal matrixes if you wish to. QED

Together with Corollary 1.7, this proves the following very important statement:

Corollary *A Euclidean motion $T : \mathbb{E}^n \to \mathbb{E}^n$ is expressed in coordinates as*

$$T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

with A an orthogonal matrix, and $\mathbf{b} \in \mathbb{R}^n$ a vector.

An immediate check shows that an orthogonal matrix A has determinant $\det A = \pm 1$ (see Lemma B.4).

Definition Let $T : \mathbb{E}^n \to \mathbb{E}^n$ be a motion expressed in coordinates as $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$. I call T *direct* (or *orientation preserving*) if $\det A = 1$ and *opposite* (or *orientation reversing*) if $\det A = -1$.

The meaning of this notion in $\mathbb{E}^2$ and $\mathbb{E}^3$ is familiar in terms of left–right orientation, and it may seem pretty intuitive that it does not depend on the choice of coordinates. However, I leave the proof to Exercise 6.8.

1.11 Normal form of an orthogonal matrix

The point of this section is to express an orthogonal map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ in a simple form in a suitable orthonormal basis of $\mathbb{R}^n$. This section may seem an obscure digression into linear algebra, but the result is central to understanding motions of Euclidean space.

1.11.1 The 2×2 rotation and reflection matrixes As a prelude to an attack on the general problem, consider the instructive case $n = 2$. The conditions for a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to be orthogonal are:

$${}^{t}AA = 1 \iff \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \begin{cases} a^2 + c^2 = 1 \\ ab + cd = 0 \\ b^2 + d^2 = 1. \end{cases}$$

Now $(a, c) \in \mathbb{R}^2$ is a point of the unit circle, so I can write $a = \cos\theta$, $c = \sin\theta$ for some $\theta \in [0, 2\pi)$ (Figure 1.11a). Then there are just two possibilities for b, d, giving

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$