Foliated spaces look locally like products, but their global structure is generally not a product, and tangential differential operators are correspondingly more complex. In the 1980s, Alain Connes founded what is now known as noncommutative geometry and topology. One of the first results was his generalization of the Atiyah-Singer index theorem to compute the analytic index associated with a tangential (pseudo)-differential operator and an invariant transverse measure on a foliated manifold, in terms of topological data on the manifold and the operator.

This book presents a complete proof of this beautiful result, generalized to foliated spaces (not just manifolds). It includes the necessary background from analysis, geometry, and topology. This second edition has improved exposition, an updated bibliography, an index, and additional material covering developments and applications since the first edition came out, including the confirmation of the Gap Labeling Conjecture of Jean Bellissard.

Mathematical Sciences Research Institute Publications

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Global Analysis on Foliated Spaces

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Second Edition

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To Doris and Rivka

Contents

Preface to the Second Edition

A lot has happened in the realm of foliated spaces and their operator algebras since 1988, when this book first appeared. We are pleased that, as we had hoped, this book has served as an introduction to the subject and a reference for researchers and students.

Our colleagues have convinced us that there is merit in issuing a second edition of our work, so that a new generation of students may have access to its contents. Cambridge University Press was amenable to the idea, so we (slowly) went to work.

We have taken the opportunity of a new edition to make a number of changes and additions to the book:

(1) We have corrected a few minor errors, filled some gaps, and made many changes to improve the exposition.

(2) We have added updates at the end of each chapter as well as occasional footnotes in which we discuss some of the relevant mathematical developments since 1988. This discussion is understandably brief. We do try to point the reader to the papers where the results themselves appear.

(3) We have enlarged the bibliography correspondingly.

(4) We have added a new appendix; it is a reprint of a *Mathematical Reviews* Featured Review by the second author on the Gap Labeling Theorem. We felt this was appropriate since it illustrates a very interesting and important application of the Index Theorem.

(5) We have added an index to the book.

(6) MSRI has provided for the resetting of the book in LaTeX and for the redrafting of all the art.

As originally formulated, the Connes' Index Theorem [1979] applied to foliated manifolds. The version presented here is valid for foliated spaces, a category that is strictly larger than foliated manifolds and laminations obtained from manifolds. It turns out that this extra generality is crucial for some of the applications of the Index Theorem in the past few years. For instance, the

Gap Labeling results discussed in Appendix D require this extra generality. We discuss this in some detail at the end of Chapter VIII.

We acknowledge with gratitude the help that we have received from Jean Bellissard, Alberto Candel, Larry Conlon, Steve Hurder, Jerry Kaminker, Masoud Khalkhali, Paul Muhly, and especially our friend and editor *par excellence* Silvio Levy in the preparation of this edition. The second author is grateful to Baruch Solel and the faculty of the Technion for a sabbatical year at a critical time. We are grateful to the editors of *Mathematical Reviews* for permission to reproduce the Featured Review on the Gap Labeling Theorem as Appendix D of this work.

Finally, we can but repeat the final paragraph of the preface of the first edition: *We owe a profound debt to Alain Connes, whose work on the index theorem aroused our own interest in the subject. This work would not exist had we not been so stimulated by his results to try to understand them better.*

Preface to the First Edition

This book grew out of lectures and the lecture notes generated therefrom by the first named author at UC Berkeley in 1980 and by the second named author at UCLA, also in 1980. We were motivated to develop these notes more fully by the urgings of our colleagues and friends and by the desire to make the general subject and the work of Alain Connes in particular more readily accessible to the mathematical public. The book develops a variety of aspects of analysis and geometry on foliated spaces which should be useful in many contexts. These strands are then brought together to provide a context and to expose Connes' index theorem for foliated spaces [Connes 1979], a theorem which asserts the equality of the analytic and the topological index (two real numbers) which are associated to a tangentially elliptic operator. The exposition, we believe, serves an additional purpose of preparing the way towards the more general index theorem of Connes and Skandalis [1981; 1984]. This index theorem describes the abstract index class in $K_0(C_r^*(G(M)))$, the index group of the C^*-algebra of the foliated space, and is necessarily substantially more abstract, while the tools used here are relatively elementary and straightforward, and are based on the heat equation method.

We must thank several people who have aided us in the preparation of this book. The origins of this book are embedded in lectures and seminars at Berkeley and UCLA (respectively) and we wish to acknowledge the patience and assistance of our colleagues there, particularly Bill Arveson, Ed Effros, Marc Rieffel and Masamichi Takesaki. More recently, we have benefitted from conversations and help from Ron Douglas, Peter Gilkey, Jane Hawkins, Steve Hurder, Jerry Kaminker, John Roe, Jon Rosenberg, Bert Schreiber, George Skandalis, Michael Taylor, and Bob Zimmer.

We owe a profound debt to Alain Connes, whose work on the index theorem aroused our own interest in the subject. This work would not exist had we not been so stimulated by his results to try to understand them better.