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# Double Affine Hecke Algebras

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Dedicated to Ian Macdonald

## PREFACE

This book is based on a series of lectures delivered by the author in Kyoto in 1996–97 and at Harvard University in 2001. The first chapter was written in collaboration with T. Akasaka, E. Date, K. Iohara, M. Jimbo, M. Kashiwara, T. Miwa, M. Noumi, Y. Saito, and K. Takemura. V. Ostrik is the coauthor of the second chapter. The author owes them a lot, as well as P. Etingof, D. Kazhdan, M. Nazarov, and E. Opdam for help and encouragement. The book was supported in part by the National Science Foundation and the Clay Mathematics Institute.

In many ways, this book began with one man, Ian Macdonald. I am deeply indebted to him.

After a comprehensive introduction, the classical and quantum Knizhnik–Zamolodchikov equations attached to root systems are studied, and their relations to the affine Hecke algebras, Kac–Moody algebras, and harmonic analysis discussed. These equations are of key importance in the analytic theory of Coxeter groups.

In Chapter 2, we switch to a systematic theory of the one-dimensional double affine Hecke algebra and its representations. It is the simplest case, but far from trivial. This algebra is closely connected with  $\mathfrak{sl}_2$ ,  $\widehat{\mathfrak{sl}}_2$ , the Heisenberg and Weyl algebras, and has impressive applications.

The third chapter is about DAHA in full generality, including the Macdonald polynomials, Fourier transform, Gauss–Selberg integrals, Verlinde algebras, Gaussian sums, and diagonal coinvariants. The transition to this abstract level will be smooth for readers familiar with root systems. Only reduced root systems are considered.

This book is essentially self-contained. The chapters are relatively independent. I hope that it will be helpful for both mathematicians and physicists who want to master the new double Hecke algebra technique.

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