# Chapter 0

# Introduction

## 0.0 Universality of Hecke algebras

### 0.0.1 Real and imaginary

Before a systematic exposition, I will try to outline the connections of the representation theory of Lie groups, Lie algebras, and Kac–Moody algebras with the Hecke algebras and the Macdonald theory.

The development of mathematics may be illustrated by Figure 0.1.

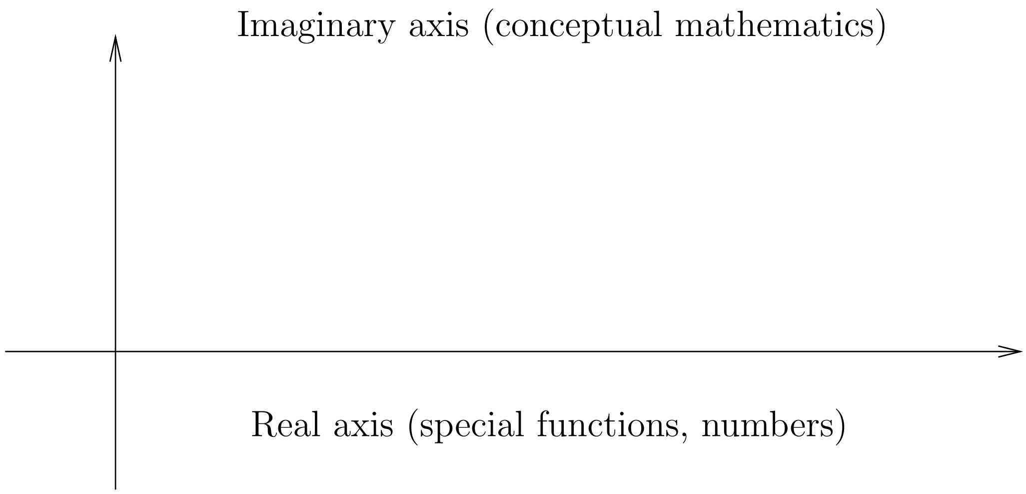


Figure 0.1: Real and Imaginary

Mathematics is fast in the imaginary (conceptual) direction but, generally, slow in the real direction (fundamental objects). Mainly I mean modern mathematics, but it may be more universal. For instance, the ancient Greeks created a highly conceptual axiomatic geometry with modest "real output." I do not think that the ratio $\Re/\Im$ is much higher now.

Let us try to project representation theory on the real axis. In Figure 0.2 we focus on Lie groups, Lie algebras, and Kac–Moody algebras, omitting the arithmetic direction (adèles and automorphic forms). The theory of special functions, arithmetic, and related combinatorics are the classical objectives of representation theory.

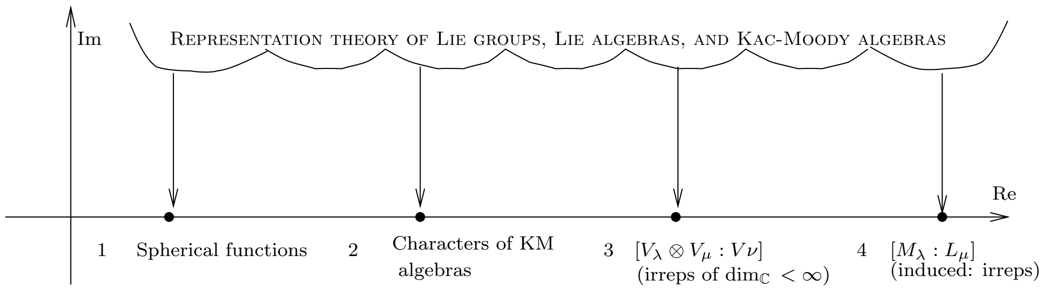


Figure 0.2: Representation Theory

Without going into detail and giving exact references, the following are brief explanations.

(1) I mean the zonal spherical functions on $K\backslash G/K$ for maximal compact $K$ in a semisimple Lie group $G$. The modern theory was started by Berezin, Gelfand, and others in the early 1950s and then developed significantly by Harish-Chandra [HC]. Lie groups greatly helped to make the classical theory multidimensional, although they did not prove to be very useful for the hypergeometric function.

(2) The characters of Kac–Moody (KM) algebras are not far from the products of classical one-dimensional $\theta$–functions and can be introduced without representation theory (Looijenga, Kac, Saito). See [Lo]. However, it is a new and important class of special functions with various applications. Representation theory explains some of their properties, but not all.

(3) This arrow gives many combinatorial formulas. Decomposing tensor products of finite dimensional representations of compact Lie groups and related problems were the focus of representation theory in the 1970s and early 1980s. They are still important, but representation theory moved toward infinite dimensional objects.

(4) Calculating the multiplicities of irreducible representations of Lie algebras in the BGG–Verma modules or other induced representations belongs to conceptual mathematics. The Verma modules were designed

as a technical tool for the Weyl character formula (which is “real”). It took time to understand that these multiplicities are “real” too, with strong analytic and modular aspects.

### 0.0.2 New vintage

Figure 0.3 is an update of Figure 0.2. We add the results which were obtained in the 1980s and 1990s, inspired mainly by a breakthrough in mathematical physics, although mathematicians had their own strong reasons to study generalized hypergeometric functions and modular representations.

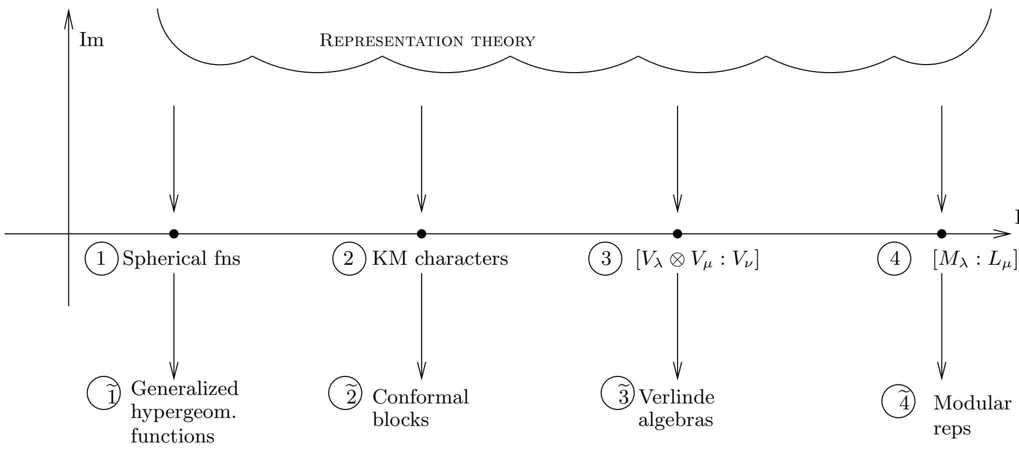


Figure 0.3: New Vintage

($\tilde{1}$) These functions will be the main subject of the first chapter. We will study them in the differential and difference cases. The interpretation and generalization of the hypergeometric functions via representation theory was an important problem of the so-called Gelfand program and remained unsolved for almost three decades.

($\tilde{2}$) Actually, the conformal blocks belong to the (conceptual) imaginary axis as well as their kin, the $\tau$–function. However, they extend the hypergeometric functions, theta functions, and Selberg’s integrals. They attach the hypergeometric function to representation theory, but affine Hecke algebras serve this purpose better.

($\tilde{3}$) The Verlinde algebras were born from conformal field theory. They are formed by integrable representations of Kac–Moody algebras of a given level with “fusion” instead of tensoring. These algebras can be also defined using quantum groups at roots of unity and interpreted via operator algebras.

($\tilde{4}$) Whatever you may think about the "reality" of $[M_\lambda : L_\mu]$, these multiplicities are connected with the representations of Lie groups and Weyl groups over finite fields (modular representations). Nothing can be more real than finite fields!

### 0.0.3 Hecke algebras

The Hecke operators and later the Hecke algebras were introduced in the theory of modular forms, actually in the theory of $GL_2$ over the $p$–adic numbers. In spite of their $p$–adic origin, they appeared to be directly connected with the $K$–theory of the *complex* flag varieties [KL1] and, more recently, with the Harish-Chandra theory. It suggests that finite and $p$–adic fields are of greater fundamental importance for mathematics and physics than we think.

Concerning the great potential of $p$–adics, let me mention the following three well-known confirmations:

(i) The Kubota–Leopold $p$–adic zeta function, which is a $p$–adic analytic continuation of the values of the classical Riemann zeta function at negative integers.
(ii) My theorem about "switching local invariants" based on the $p$–adic uniformization (Tate–Mumford) of the modular curves which come from the quaternion algebras.
(iii) The theory of $p$–adic strings due to Witten, which is based on the similarity of the Frobenius automorphism in arithmetic to the Dirac operator.

**Observation.** *The real projection of representation theory goes through Hecke-type algebras.*

The arrows in Figure 0.4 are as follows.

($a$) This arrow is the most recognized now. Quite a few aspects of the Harish-Chandra theory in the zonal case were covered by representation theory of the degenerate (graded) affine Hecke algebras, introduced in [Lus1] ([Dr1] for $GL_n$). The radial parts of the invariant differential operators on symmetric spaces, the hypergeometric functions and their generalizations arise directly from these algebras [C10].

The difference theory appeared even more promising. It was demonstrated in [C19] that the $q$–Fourier transform is self-dual like the classical Fourier and Hankel transforms, but not the Harish-Chandra transform. There are connections with the quantum groups and quantum

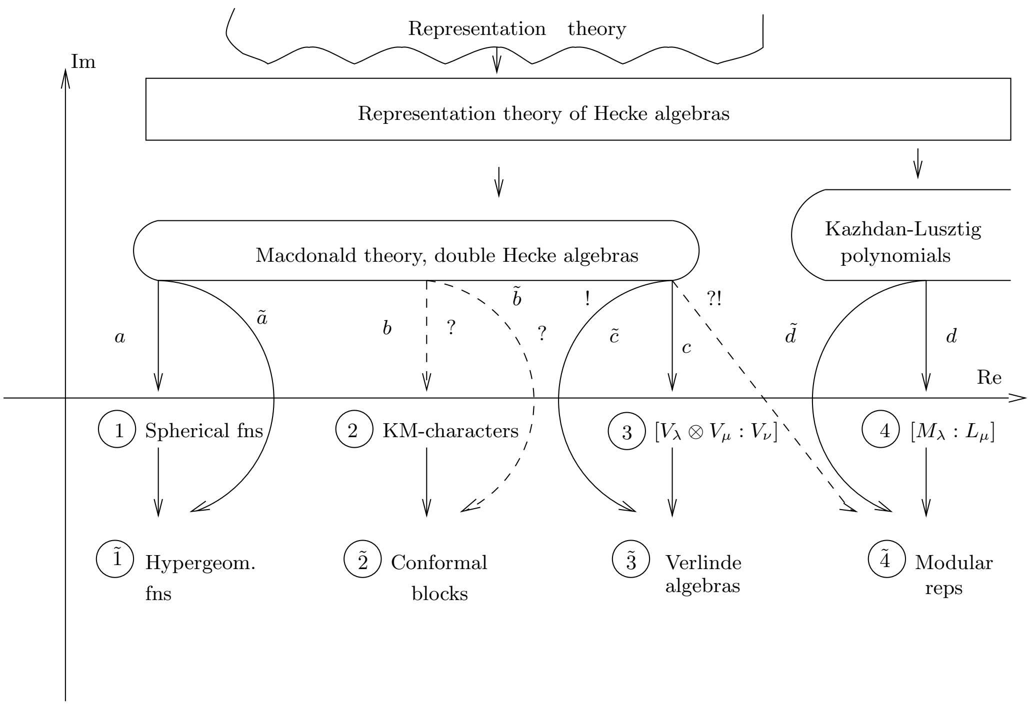


Figure 0.4: Hecke Algebras

symmetric spaces (Noumi, Olshansky, and others; see [No1]). However, the double Hecke algebra technique is simpler and more powerful.

($b$) The conformal blocks are solutions of the KZ–Bernard equation (KZB). The double Hecke algebras lead to certain elliptic generalizations of the Macdonald polynomials [C17, C18, C23] (other approaches are in [EK1, C17, FV3], and the recent [Ra]). These algebras govern the monodromy of the KZB equation and "elliptic" Dunkl operators (Kirillov Jr., Felder–Tarasov–Varchenko, and the author).

The monodromy map is the inverse of arrow ($\tilde{b}$). The simplest examples are directly related to the Macdonald polynomials and those at roots of unity.

($c$) Hecke algebras and their affine generalizations give a new approach to the classical combinatorics, including the characters of the compact Lie groups. The natural setting here is the theory of the Macdonald polynomials, although the analytic theory seems more challenging.

Concerning ($\tilde{c}$), the Macdonald polynomials at the roots of unity give a simple approach to the Verlinde algebras [Ki1, C19, C20]. The use of the nonsymmetric Macdonald polynomials here is an important development. Generally, these polynomials are beyond the Lie and Kac–Moody theory, although they are connected with the Heisenberg–Weyl and $p$–adic Hecke algebras.

($d$) This arrow is the Kazhdan–Lusztig conjecture proved by Brylinski–Kashiwara and Beilinson–Bernstein and then generalized to the Kac–Moody case by Kashiwara–Tanisaki.

By ($\tilde{d}$), I mean the modular Lusztig conjecture (partially) proved by Anderson, Jantzen, and Soergel. There is recent significant progress due to Bezrukavnikov.

The arrow from the Macdonald theory to modular representations is marked by "?!." It seems to be the most challenging now (there are already first steps in this direction). It is equivalent to extending the Verlinde algebras and their nonsymmetric variants from the alcove (the restricted category of representations of Lusztig's quantum group) to the parallelogram (all representations).

If such an extension exists, it would give a $k$–extension of Lusztig's conjectures, formulas for the modular characters (not only those for the multiplicities), a description of modular representations for arbitrary Weyl groups, and more.

## 0.1 KZ and Kac–Moody algebras

In this section we comment on the role of the Kac–Moody algebras and their relations (real and imaginary) to the spherical functions and the double Hecke algebras.

### 0.1.1 Fusion procedure

I think that the penetration of double Hecke algebras into the fusion procedure and related problems of the theory of Kac–Moody algebras is a convincing demonstration of their potential. The fusion procedure was introduced for the first time in [C3]. On the physics side, let me also mention a contribution of Louise Dolan.

Given an integrable representation of the $n$–th power of a Kac–Moody algebra and two sets of points on a Riemann surface ($n$ points and $m$ points), I constructed an integrable representation of the $m$–th power of the same Kac–Moody algebra. The construction does not change the "global" central charge, the sum of the local central charges over the components. It was named later "fusion procedure."

I missed that in the special case of this correspondence, when $n = 2$ and $m = 1$, the multiplicities of irreducibles in the resulting representation are the structural constants of a certain commutative algebra, the Verlinde algebra [Ver].

Now we know that the Verlinde algebra and all its structures can be readily extracted from the simplest representation of the double affine Hecke algebra at roots of unity. Thus the Kac–Moody algebras are undoubtedly connected with the double Hecke algebras.

Double Hecke algebras dramatically simplify and generalize the algebraic theory of Verlinde algebras, including the inner product and the (projective) action of $PSL(2, \mathbb{Z})$, however, excluding the integrality and positivity of the structural constants. The latter properties require $k = 1$ and are closely connected with the Kac–Moody interpretation (although they can also be checked directly).

I actually borrowed the fusion procedure from Y. Ihara's papers "On congruence monodromy problem." A similar construction is a foundation of his theory. I changed and added some things (the central charge has no counterpart in his theory), but the procedure is basically the same. Can we go back and define Verlinde algebras in adèles' setting?

### 0.1.2 Symmetric spaces

The classification of Kac–Moody algebras very much resembles that of symmetric spaces. See [Ka], [He2]. It is not surprising, because the key technical point in both theories is the description of the involutions and automorphisms of finite order for the semisimple finite dimensional Lie algebras. The classification lists are similar *but do not coincide.* For instance, the $BC_n$–symmetric spaces have no Kac–Moody counterparts. Conversely, the KM algebra of type, say, $D_4^{(3)}$ is not associated (even formally) with any symmetric space. Nevertheless one could hope that this parallelism is not incidental.

Some kind of correspondence can be established using the isomorphism of the quantum many-body problem [Ca, Su, HO1], a direct generalization of the Harish-Chandra theory, and the affine KZ equation. The isomorphism was found by A. Matsuo and developed further in my papers. It holds when the parameter $k$, given in terms of the root multiplicity in the context of symmetric spaces, is an arbitrary complex number. In the Harish-Chandra theory, it equals $1/2$ for $SL_2(\mathbb{R})/SO_2$, 1 in the so-called group case $SL_2(\mathbb{C})/SU_2$, and $k = 2$ for the $Sp_2$. The $k$–generalized spherical functions are mainly due to Heckman and Opdam; see Chapter 1.

Once $k$ was made an arbitrary number, it could be expected a counterpart of the central charge $c$, the level, in the theory of Kac–Moody algebras. Indeed, it has some geometric meaning. However, generally, it is *not* connected with the central charge. Indeed, the number of independent $k$–parameters can be from 1 $(A, D, E)$ to 5 ($C^\vee C$, the so-called Koornwinder case), but we have only one (global) central element $c$ in the Kac–Moody theory. Also, the $k$–spherical functions are eigenfunctions of differential operators generalizing the radial parts of the invariant operators on symmetric spaces. These operators have no counterparts for the Kac–Moody characters. Also, the spherical functions are orthogonal polynomials; the Kac–Moody charactes are not. In addition, the latter are of elliptic type, the spherical functions are of trigonometric type.

We will discuss the elliptic quantum many-body problem (QMBP) in the first chapter. It gives a kind of theory of spherical functions in the Kac–Moody setting (at critical level). However, it supports the *unification* of $c$ and $k$ rather than the *correspondence* between them.

The elliptic QMBP in the $GL_N$–case was introduced by Olshanetsky and Perelomov [OP]. The classical root systems were considered in the paper [OOS]. The Olshanetsky–Perelomov operators for arbitrary root systems were constructed in [C17].

We see that an exact match cannot be expected. However, a map from the Kac–Moody algebras to spherical functions exists. It is for $GL_N$ only and not exactly for the KM characters, but it does exist.

### 0.1.3 KZ and $r$–matrices

The KZ equation is the system of differential equations for the matrix elements (using physical terminology, the correlation functions) of the representations of the Kac–Moody algebras in the $n$–point case. The matrix elements are simpler to deal with than the characters. For instance, they satisfy differential equations with respect to the positions of the points.

The most general "integrable" case, is described by the so-called $r$–matrix Kac–Moody algebras from [C1] and the corresponding *$r$–matrix KZ equations* introduced in [C6].

It was observed in the latter paper that the classical Yang–Baxter equation can be interpreted as the compatibility of the corresponding KZ system, which dramatically enlarged the number of examples. An immediate application was a new class of KZ equations with *trigonometric* and *elliptic* dependence on the points.

It was demonstrated in [C6] that the abstract $\tau$–function, also called the coinvariant, is a generic solution of the $r$–matrix KZ with respect to the action of the Sugawara $(-1)$–operators.

More generally, the $r$–matrices and the corresponding KZ equations attached to arbitrary*root systems* were defined in [C6]. For instance, the dependence on the points is via the differences (the $A$–case) of the points and also via the *sums* for $B, C, D$. The $BC$–case is directly related to the so-called *reflection equations* introduced in [C2].

The results due to Drinfeld–Kohno on the monodromy of the KZ equations (see [Ko]) can be extended to the $r$–matrix equations. In some cases, the monodromy can be calculated explicitly, for instance, for the *affine KZ* [C6, C7, C8].

### 0.1.4 Integral formulas for KZ

The main applications of the interpretation of KZ as a system of equations for the coinvariant were: (i) a simplification of the algebraic part of the Schechtman–Varchenko construction [SV] of integral formulas for the rational KZ, (ii) a generalization of their formulas to the trigonometric case [C9]. Paper [SV] is based on direct algebraic considerations without using the theory of Kac–Moody algebras.

There is another important "integrable" case, the so-called Knizhnik–Zamolodchikov–Bernard equation usually denoted by KZB [Be, FW1]. We will see in Chapter 1 that it can be obtain in the same abstract manner as a system of differential equations for the corresponding "elliptic" coinvariant. There must be an implication of this fact toward the integral formulas for KZB, but this has not been checked so far.

We do not discuss the integral formulas for KZB in this book, as well as the integral formulas for QKZ, the quantum Knizhnik–Zamolodchikov equation. See, e.g., [TV], [FV1], and [FTV].

Generally, the KZ equations can be associated with arbitrary algebraic curves. Then they involve the derivatives with respect to the moduli of curves and vector bundles. However, in this generality, the resulting equations are non-integrable in any reasonable sense.

Summarizing, we have the following major cases, when the Knizhnik–Zamolodchikov equation have integral formulas, reasonably simple monodromy representations, special symmetries, and other important properties:
(a) the KZ for Yang's rational $r$–matrix (see [SV]),
(b) the trigonometric KZ equation introduced in [C9],
(c) the elliptic KZ–Bernard equation (see [Be, FW1]).

Given a Lie algebra $\mathfrak{g}$, one may define the integrand for the KZ integral formulas is derectly connected with the coinvariants of $U(\widehat{\mathfrak{g}})$ for the Weyl modules [C9]. The contours (cycles) of integration are governed by the quantum $U_q(\mathfrak{g})$. See [FW2], [Va] and references therein. We will not discuss the contours and the $q$–topology of the configuration spaces in this book.

The later topic was started by Aomoto [A1, AKM] and seems an endless story. We have no satisfactory formalization of the $q$–topology so far. It is especially needed for QKZ. Generally, in mathematics, the contours of integration (the homology) must be dual to the differential forms (the cohomology). It gives an approach to the problem.

We note that the integral KZ formulas are directly connected with the equivalence of the $U(\widehat{\mathfrak{g}})_c$ and the quantum group $U_q(\mathfrak{g})$ due to Kazhdan, Lusztig, and Finkelberg (see [KL2]). It is for a proper relation $c \leftrightarrow q$.

### 0.1.5 From KZ to spherical functions

Let us discuss what the integral formulas could give for the theory of spherical functions and its generalizations. There are natural limitations.

First, only the spherical functions of type $A$ may apper (for either choice of $\mathfrak{g}$) if we begin with the KZ integral formulas of type $A$.

Second, one needs an $r$–matrix KZ of *trigonometric* type because the Harish-Chandra theory is on the torus.

Third, only $\mathfrak{g} = \mathfrak{gl}_N$ may result in *scalar* differential operators due to the analysis by Etingof and Kirillov Jr.

Summarizing, the integral formulas for the affine KZ (AKZ) of type $A$ are the major candidates. The AKZ is *isomorphic* to the quantum many-body problem, that is exactly the $k$–Harish-Chandra theory [Mat, C11].

Note that the "basic" trigonometric $n$–point KZ taking values in the 0–weight component of $(\mathbb{C}^n)^{\otimes n}$, which is isomorphic to the group algebra $\mathbb{C}\mathbb{S}_n$, must be considred for AKZ. The integral AKZ formula is likely to be directly connected with the Harish-Chandra formula. I did not check it, but calculations due to Mimachi, Felder, Varchenko confirm this. For instance, the dimension of the contours (cycles) of integration for such KZ is $n(n-1)/2$, which coincides with that in the Harish-Chandra integral representation for spherical functions of type $A_{n-1}$. His integral is over $K = SO_n \subset SL_n(\mathbb{R})$.

Establishing a direct connection with the Harsh-Chandra integral representation for the spherical functions does not seem too difficult. However it is of obvious importance, because his formula is for *all* root systems, and one can use it as an initial point for the general theory of integral formulas of the KZ equations associated with root systems.

We note that the integral KZ formulas can be justified without Kac–Moody algebras. A straightforward algebraic combinatorial analysis is complicated but possible [SV]. The proof presented in this book is based on the Kac–Moody coinvariant [C9]. However, I use the Kac–Moody algebras at the critical level only, as a technical tool, and then extend the resulting formulas to all values of the center charge.

There is another approach to the same integral formulas based on the coinvariant for the Wakimoto modules instead of that for the Weyl modules [FFR]. The calculations with the coinvariant are in fact similar to mine, but