# Numerical Solution of Partial Differential Equations

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## *An Introduction*

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Second Edition

Cambridge
University Press

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK

www.cambridge.org
Information on this title: www.cambridge.org/9780521607930



First published 2005

Printed in the United Kingdom at the University Press, Cambridge

*A catalogue record for this book is available from the British Library*

ISBN-13 978-0-521-60793-0 paperback
ISBN-10 0-521-60793-0 paperback

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Cambridge University Press
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# Preface to the first edition

The origin of this book was a sixteen-lecture course that each of us has given over the last several years to final-year Oxford undergraduate mathematicians; and its development owes much to the suggestions of some of our colleagues that the subject matter could readily be taught somewhat earlier as a companion course to one introducing the theory of partial differential equations. On the other hand, we have used much of the same material in teaching a one-year Master's course on mathematical modelling and numerical analysis. These two influences have guided our choice of topics and the level and manner of presentation.

Thus we concentrate on finite difference methods and their application to standard model problems. This allows the methods to be couched in simple terms while at the same time treating such concepts as stability and convergence with a reasonable degree of mathematical rigour. In a more advanced text, or one with greater emphasis on the finite element method, it would have been natural and convenient to use standard Sobolev space norms. We have avoided this temptation and used only discrete norms, specifically the maximum and the $l_2$ norms. There are several reasons for this decision. Firstly, of course, it is consistent with an aim of demanding the minimum in prerequisites – of analysis, of PDE theory, or of computing – so allowing the book to be used as a text in an early undergraduate course and for teaching scientists and engineers as well as mathematicians.

Equally importantly though, the decision fits in with our widespread use of discrete maximum principles in analysing methods for elliptic and parabolic problems, our treatment of discrete energy methods and conservation principles, and the study of discrete Fourier modes on finite domains. We believe that treating all these ideas at a purely discrete level helps to strengthen the student's understanding of these important mathematical tools. At the same time this is a very practical approach, and it encourages the interpretation of difference schemes as direct models of physical principles and phenomena: all calculations are, after all, carried out on a finite grid, and practical computations are checked for stability, etc. at the discrete level. Moreover, interpreting a difference scheme's effect on the Fourier modes that can be represented on the mesh, in terms of the damping and dispersion in one time step is often of greater value than considering the truncation error, which exemplifies the second justification of our approach.

However, the limiting process as a typical mesh parameter $h$ tends to zero is vital to a proper understanding of numerical methods for partial differential equations. For example, if $U^n$ is a discrete approximation at time level $n$ and evolution through a time step $\Delta t$ is represented as $U^{n+1} = C_h U^n$, many students find great difficulty in distinguishing the limiting process when $n \to \infty$ on a fixed mesh and with fixed $\Delta t$, from that in which $n \to \infty$ with $n\Delta t$ fixed and $h, \Delta t \to 0$. Both processes are of great practical importance: the former is related to the many iterative procedures that have been developed for solving the discrete equations approximating steady state problems by using the analogy of time stepping the unsteady problem; and understanding the latter is crucial to avoiding instability when choosing methods for approximating the unsteady problems themselves. The notions of uniform bounds and uniform convergence lie at the heart of the matter; and, of course, it is easier to deal with these by using norms which do not themselves depend on $h$. However, as shown for example by Palencia and Sanz–Serna,[1] a rigorous theory can be based on the use of discrete norms and this lies behind the approach we have adopted. It means that concepts such as well-posedness have to be rather carefully defined; but we believe the slight awkwardness entailed here is more than compensated for by the practical and pedagogical advantages pointed out above.

The ordering of the topics is deliberate and reflects the above concerns. We start with parabolic problems, which are both the simplest to approximate and analyse and also of widest utility. Through the addition of convection to the diffusion operator, this leads naturally to the study of hyperbolic problems. It is only after both these cases have been explored in some detail that, in Chapter 5, we present a careful treatment of the concepts of consistency, convergence and stability for evolutionary problems. The final two chapters are devoted respectively

[1] Palencia, C. and Sanz–Serna, J. M. (1984), An extension of the Lax–Richtmyer theory, *Numer. Math.* **44** (2), 279–283.

to the discretisation of elliptic problems, with a brief introduction to finite element methods, and to the iterative solution of the resulting algebraic equations; with the strong relationship between the latter and the solution of parabolic problems, the loop of linked topics is complete. In all cases, we present only a small number of methods, each of which is thoroughly analysed and whose practical utility we can attest to. Indeed, we have taken as a theme for the book that all the model problems and the methods used to approximate them are simple but generic.

Exercises of varying difficulty are given at the end of each chapter; they complete, extend or merely illustrate the text. They are all analytical in character, so the whole book could be used for a course which is purely theoretical. However, numerical analysis has very practical objectives, so there are many numerical illustrations of the methods given in the text; and further numerical experiments can readily be generated for students by following these examples. Computing facilities and practices develop so rapidly that we believe this open-ended approach is preferable to giving explicit practical exercises.

We have referred to the relevant literature in two ways. Where key ideas are introduced in the text and they are associated with specific original publications, full references are given in footnotes – as earlier in this Preface. In addition, at the end of each chapter we have included a brief section entitled 'Bibliographic notes and recommended reading' and the accumulated references are given at the end of the book. Neither of these sets of references is intended to be comprehensive, but they should enable interested students to pursue further their studies of the subject. We have, of course, been greatly guided and influenced by the treatment of evolutionary problems in Richtmyer and Morton (1967); in a sense the present book can be regarded as both introductory to and complementary to that text.

We are grateful to several of our colleagues for reading and commenting on early versions of the book (with Endre Süli's remarks being particularly helpful) and to many of our students for checking the exercises. The care and patience of our secretaries Jane Ellory and Joan Himpson over the long period of the book's development have above all made its completion possible.

# Preface to the second edition

In the ten years since the first edition of this book was published, the numerical solution of PDEs has moved forward in many ways. But when we sought views on the main changes that should be made for this second edition, the general response was that we should not change the main thrust of the book or make very substantial changes. We therefore aimed to limit ourselves to adding no more than 10%–20% of new material and removing rather little of the original text: in the event, the book has increased by some 23%.

Finite difference methods remain the starting point for introducing most people to the solution of PDEs, both theoretically and as a tool for solving practical problems. So they still form the core of the book. But of course finite element methods dominate the elliptic equation scene, and finite volume methods are the preferred approach to the approximation of many hyperbolic problems. Moreover, the latter formulation also forms a valuable bridge between the two main methodologies. Thus we have introduced a new section on this topic in Chapter 4; and this has also enabled us to reinterpret standard difference schemes such as the Lax–Wendroff method and the box scheme in this way, and hence for example show how they are simply extended to nonuniform meshes. In addition, the finite element section in Chapter 6 has been followed by a new section on convection–diffusion problems: this covers both finite difference and finite element schemes and leads to the introduction of Petrov–Galerkin methods.

The theoretical framework for finite difference methods has been well established now for some time and has needed little revision. However, over the last few years there has been greater interaction between methods to approximate ODEs and those for PDEs, and we have responded to this stimulus in several ways. Firstly, the growing interest in applying symplectic methods to Hamiltonian ODE systems, and extending the approach to PDEs, has led to our including a section on this topic in Chapter 4 and applying the ideas to the analysis of the staggered leap–frog scheme used to approximate the system wave equation. More generally, the revived interest in the method of lines approach has prompted a complete redraft of the section on the energy method of stability analysis in Chapter 5, with important improvements in overall coherence as well as in the analysis of particular cases. In that chapter, too, is a new section on modified equation analysis: this technique was introduced thirty years ago, but improved interpretations of the approach for such as the box scheme have encouraged a reassessment of its position; moreover, it is again the case that its use for ODE approximations has both led to a strengthening of its analysis and a wider appreciation of its importance.

Much greater changes to our field have occurred in the practical application of the methods we have described. And, as we continue to have as our aim that the methods presented should properly represent and introduce what is used in practice, we have tried to reflect these changes in this new edition. In particular, there has been a huge improvement in methods for the iterative solution of large systems of algebraic equations. This has led to a much greater use of implicit methods for time-dependent problems, the widespread replacement of direct methods by iterative methods in finite element modelling of elliptic problems, and a closer interaction between the methods used for the two problem types. The emphasis of Chapter 7 has therefore been changed and two major sections added. These introduce the key topics of multigrid methods and conjugate gradient methods, which have together been largely responsible for these changes in practical computations.

We gave serious consideration to the possibility of including a number of MATLAB programs implementing and illustrating some of the key methods. However, when we considered how very much more powerful both personal computers and their software have become over the last ten years, we realised that such material would soon be considered outmoded and have therefore left this aspect of the book unchanged. We have also dealt with references to the literature and bibliographic notes in the same way as in the earlier edition: however, we have collected both into the reference list at the end of the book.

Solutions to the exercises at the end of each chapter are available in the form of LaTeX files. Those involved in teaching courses in this area can obtain copies, by email only, by applying to solutions@cambridge.org.

We are grateful to all those readers who have informed us of errors in the first edition. We hope we have corrected all of these and not introduced too many new ones. Once again we are grateful to our colleagues for reading and commenting on the new material.