

## Modeling Aggregate Behavior and Fluctuations in Economics

This book has two components: stochastic dynamics and stochastic random combinatorial analysis. The first discusses evolving patterns of interactions of a large but finite number of agents of several types. Changes of agent types or their choices or decisions over time are formulated as jump Markov processes with suitably specified transition rates: Optimizations by agents make these rates generally endogenous. Probabilistic equilibrium selection rules are also discussed, together with the distributions of relative sizes of the basin of attraction. As the number of agents approaches infinity, we recover deterministic macroeconomic relations of more conventional economic models. The second component analyzes how agents form clusters of various sizes. This has applications for discussing sizes or shares of markets by several types of agents, which involves some combinatorial analysis patterned after the population genetics literature. These are shown to be relevant to distributions of returns to assets, volatility of returns, and power laws.

Masanao Aoki is Professor Emeritus in the Department of Economics at the University of California, Los Angeles. He has held professorial appointments at the Institute for Social and Economic Research at Osaka University, Tokyo Institute of Technology, and the University of Illinois. Professor Aoki is a past President of the Society for Economic Dynamics and Control, a Fellow of the Econometric Society, and a Fellow of the IEEE Control Systems Society. Currently Associate Editor of the journal *Macroeconomic Dynamics*, published by Cambridge University Press, he also served as Editor of the *Journal of Economic Dynamics and Control* and the *International Economic Review* and as Associate Editor of the IEEE's *Transactions of Automatic Control, Information Sciences*, and the *Journal of Mathematical Analysis and Application*. Professor Aoki is the author or editor of a dozen books, including *New Approaches to Macroeconomic Modeling: Evolutionary Stochastic Dynamics, Multiple Equilibria, and Externalities as Field Effects* (Cambridge University Press, 1996).

# Modeling Aggregate Behavior and Fluctuations in Economics

**Stochastic Views of Interacting Agents**

MASANAO AOKI  
*University of California, Los Angeles*



**CAMBRIDGE**  
UNIVERSITY PRESS

**CAMBRIDGE**  
 UNIVERSITY PRESS

32 Avenue of the Americas, New York NY 10013-2473, USA

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9780521606196](http://www.cambridge.org/9780521606196)

© Masanao Aoki 2002

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2002

First paperback edition 2004

*A catalogue record for this publication is available from the British Library*

*Library of Congress Cataloguing in Publication data*

Aoki, Masanao.

Modeling aggregate behavior and fluctuations in economics : stochastic views of interacting agents / Masanao Aoki.

p. cm.

Includes bibliographical references and index.

ISBN 0-521-78126-4

1. Demand (Economic theory) – Mathematical models. 2. Supply and demand – Mathematical models. 3. Consumption (Economics) – Mathematical models.
4. Business cycles – Mathematical models. 5. Statics and dynamics (Social sciences) – Mathematical models. 6. Stochastic processes – Mathematical models. I. Title.

HB842 .A57 2001

338.5212 – dc21

2001025596

ISBN 978-0-521-60619-6 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Cambridge University Press  
978-0-521-60619-6 — Modeling Aggregate Behavior and Fluctuations in Economics  
Masanao Aoki  
Frontmatter  
[More Information](#)

---

*To My Late Father*

## CONTENTS

<b>Preface</b>	<b>xiii</b>
<b>1 Overviews</b>	<b>1</b>
1.1 Our Objectives and Approaches	1
1.2 Partial List of Applications	2
1.3 States: Vectors of Fractions of Types and Partition Vectors	3
1.3.1 Vectors of Fractions	4
1.3.2 Partition Vectors	5
1.4 Jump Markov Processes	6
1.5 The Master Equation	7
1.6 Decomposable Random Combinatorial Structures	8
1.7 Sizes and Limit Behavior of Large Fractions	8
<b>2 Setting Up Dynamic Models</b>	<b>9</b>
2.1 Two Kinds of State Vectors	10
2.2 Empirical Distributions	11
2.3 Exchangeable Random Sequences	12
2.4 Partition Exchangeability	13
2.5 Transition Rates	16
2.6 Detailed-Balance Conditions and Stationary Distributions	17
<b>3 The Master Equation</b>	<b>19</b>
3.1 Continuous-Time Dynamics	19
3.2 Power-Series Expansion	23
3.3 Aggregate Dynamics and Fokker–Planck Equation	25
3.4 Discrete-Time Dynamics	25
<b>4 Introductory Simple and Simplified Models</b>	<b>27</b>
4.1 A Two-Sector Model of Fluctuations	27
	<b>vii</b>

viii	<b>Contents</b>	
	4.2 Closed Binary Choice Models	30
	4.2.1 A Pólya Distribution Model	31
	4.3 Open Binary Models	32
	4.3.1 Examples	35
	4.4 Two Logistic Process Models	35
	4.4.1 Model 1: The Aggregate Dynamics and Associated Fluctuations	35
	4.4.2 Model 2: Nonlinear Exit Rate	37
	4.4.3 A Nonstationary Pólya Model	38
	4.5 An Example: A Deterministic Analysis of Nonlinear Effects May Mislead!	40
	<b>5 Aggregate Dynamics and Fluctuations of Simple Models</b>	<b>41</b>
	5.1 Dynamics of Binary Choice Models	41
	5.2 Dynamics for the Aggregate Variable	43
	5.3 Potentials	45
	5.4 Critical Points and Hazard Function	47
	5.5 Multiplicity – An Aspect of Random Combinatorial Features	49
	<b>6 Evaluating Alternatives</b>	<b>52</b>
	6.1 Representation of Relative Merits of Alternatives	53
	6.2 Value Functions	54
	6.3 Extreme Distributions and Gibbs Distributions	57
	6.3.1 Type I: Extreme Distribution	58
	6.4 Approximate Evaluations of Value Functions with a Large Number of Alternatives	60
	6.5 Case of Small Entry and Exit Probabilities: An Example	60
	6.6 Approximate Evaluation of Sums of a Large Number of Terms	61
	6.7 Approximations of Error Functions	62
	6.7.1 Generalization	64
	6.7.2 Example	65
	<b>7 Solving Nonstationary Master Equations</b>	<b>66</b>
	7.1 Example: Open Models with Two Types of Agents	66
	7.1.1 Equilibrium Distribution	67
	7.1.2 Probability-Generating-Function Method	67
	7.1.3 Cumulant Generating Functions	68
	7.2 Example: A Birth–Death-with-Immigration Process	69
	7.2.1 Stationary Probability Distribution	70
	7.2.2 Generating Function	70

<b>Contents</b>	ix
7.2.3 Time-Inhomogeneous Transition Rates	73
7.2.4 The Cumulant-Generating-Function	74
7.3 Models for Market Shares by Imitation or Innovation	75
7.3.1 Deterministic Innovation Process	76
7.3.2 Deterministic Imitation Process	77
7.3.3 A Joint Deterministic Process	78
7.3.4 A Stochastic Dynamic Model	79
7.4 A Stochastic Model with Innovators and Imitators	80
7.4.1 Case of a Finite Total Number of Firms	83
7.5 Symmetric Interactions	84
7.5.1 Stationary State Distribution	84
7.5.2 Nonstationary Distributions	84
<b>8 Growth and Fluctuations</b>	<b>85</b>
8.1 Two Simple Models for the Emergence of New Goods	87
8.1.1 Poisson Growth Model	87
8.1.2 An Urn Model for Growth	88
8.2 Disappearance of Goods from Markets	90
8.2.1 Model	91
8.2.2 Stability Analysis	92
8.3 Shares of Dated Final Goods Among Households	93
8.3.1 Model	94
8.4 Deterministic Share Dynamics	95
8.5 Stochastic Business-Cycle Model	96
8.6 A New Model of Fluctuations and Growth: Case with Underutilized Factor of Production	99
8.6.1 The Model	100
8.6.2 Transition-Rate Specifications	101
8.6.3 Holding Times	102
8.6.4 Aggregate Outputs and Demands	103
8.6.5 Equilibrium Sizes of the Sectors (Excess Demand Conditions)	104
8.6.6 Behavior Out of Equilibrium: Two-Sector Model	105
8.6.7 Stationary Probability Distribution: The Two-Sector Model	107
8.6.8 Emergence of New Sectors	110
8.6.9 Simulation Runs for Multi-Sector Model	111
8.6.10 Discussion	112
8.7 Langevin-Equation Approach	117
8.7.1 Stationary Density Function	119
8.7.2 The Exponential Distribution of the Growth Rates of Firms	119

8.8	Time-Dependent Density and Heat Equation	121
8.9	Size Distribution for Old and New Goods	122
8.9.1	Diffusion-Equation Approximation	122
8.9.2	Lines of Product Developments and Inventions	123
<b>9</b>	<b>A New Look at the Diamond Search Model</b>	<b>127</b>
9.1	Model	129
9.2	Transition Rates	129
9.3	Aggregate Dynamics: Dynamics for the Mean of the Fraction	130
9.4	Dynamics for the Fluctuations	131
9.5	Value Functions	132
9.5.1	Expected-Value Functions	133
9.6	Multiple Equilibria and Cycles: An Example	134
9.6.1	Asymmetrical Cycles	136
9.6.1.1	Approximate Analysis	136
9.6.1.2	Example	137
9.7	Equilibrium Selection	138
9.8	Possible Extensions of the Model	139
<b>10</b>	<b>Interaction Patterns and Cluster Size Distributions</b>	<b>141</b>
10.1	Clustering Processes	141
10.2	Three Classes of Transition Rates	144
10.2.1	Selections	144
10.2.2	Multisets	146
10.2.2.1	Capacity-Limited Processes	150
10.2.3	Assemblies	150
10.2.3.1	Internal Configurations of Assemblies	151
10.3	Transition-Rate Specifications in a Partition Vector	153
10.4	Logarithmic Series Distribution	153
10.4.1	Frequency Spectrum of the Logarithmic Series Distribution	156
10.5	Dynamics of Clustering Processes	157
10.5.1	Examples of Clustering-Process Distributions	157
10.5.2	Example: Ewens Sampling Formula	162
10.5.3	Dynamics of Partition Patterns: Example	164
10.6	Large Clusters	165
10.6.1	Expected Value of the Largest Cluster Size	166
10.6.2	Joint Probability Density for the Largest $r$ Fractions	169
10.7	Moment Calculations	171
10.8	Frequency Spectrum	172
10.8.1	Definition	173



<b>Contents</b>	xi
10.8.2 Herfindahl Index of Concentration	173
10.8.3 A Heuristic Derivation	174
10.8.4 Recursion Relations	176
10.8.5 Examples of Applications	177
10.8.6 Discrete Frequency Spectrum	177
10.9 Parameter Estimation	178
<b>11 Share Market with Two Dominant Groups of Traders</b>	<b>180</b>
11.1 Transition Rates	181
11.2 Ewens Distribution	183
11.2.1 The Number of Clusters and Value of $\theta$	184
11.2.2 Expected Values of the Fractions	185
11.2.3 The Largest Two Shares	186
11.3 Market Volatility	187
11.4 Behavior of Market Excess Demand	188
11.4.1 Conditions for Zero Excess Demand	188
11.4.2 Volatility of the Market Excess Demand	189
11.4.3 Approximate Dynamics for Price Differences and Power Law	190
11.4.3.1 Simulation Experiments	192
<b>Appendix</b>	<b>195</b>
A.1 Deriving Generating Functions via Characteristic Curves	195
A.2 Urn Models and Associated Markov Chains	197
A.2.1 Pólya's Urn	197
A.2.2 Hoppe's Urn	197
A.2.3 Markov Chain Associated with the Urn	199
A.3 Conditional Probabilities for Entries, Exits, and Changes of Type	200
A.3.1 Transition Probabilities	200
A.3.2 Transition Rates	202
A.4 Holding Times and Skeletal Markov Chains	202
A.4.1 Sojourn-Time Models	205
A.5 Stirling Numbers	206
A.5.1 Introduction	206
A.5.2 Recursions	207
A.5.3 Relations with Combinatorics	209
A.5.4 Explicit Expressions and Asymptotic Relations	210
A.5.5 Asymptotics	212
A.6 Order Statistics	213
A.7 Poisson Random Variables and the Ewens Sampling Formula	214

xii	<b>Contents</b>	
	A.7.1	Approximations by Poisson Random Variables 214
	A.7.2	Conditional Poisson Random Variables 216
A.8		Exchangeable Random Partitions 219
	A.8.1	Exchangeable Random Sequences 219
	A.8.2	Partition Exchangeability 221
A.9		Random Partitions and Permutations 224
	A.9.1	Permutations 224
	A.9.2	Random Partitions 225
	A.9.3	Noninterference of Partitions 228
	A.9.4	Consistency 229
A.10		Dirichlet Distributions 229
	A.10.1	Beta Distribution 229
	A.10.2	Dirichlet Distribution 230
	A.10.3	Marginal Dirichlet Distributions 232
	A.10.4	Poisson–Dirichlet Distribution 232
	A.10.5	Size-Biased Sampling 233
A.11		Residual Allocation Models 234
A.12		GEM and Size-Biased Distributions 235
A.13		Stochastic Difference Equations 240
A.14		Random Growth Processes 242
A.15		Diffusion Approximation to Growth Processes 243
	<b>References</b>	<b>245</b>
	<b>Index</b>	<b>253</b>

## PREFACE

This book is a sequel to Aoki (1996) in the loose sense that it is motivated by a similar set of considerations to its predecessor and shares some of the same objectives. It records my efforts, since the publication of that book, at evaluating and reformulating macroeconomic models that are employed by the mainstream economic profession. In this book, a stochastic point of view is taken to construct models for finite numbers of interacting agents. In other words, the book emphasizes models that focus on economic phenomena that involve stochastic laws, or stochastic regularities that govern economic phenomena.

To make this book more readily accessible to traditionally trained economists and graduate students in economics, it is more narrowly focused than my previous one, and it attempts to establish better links with some well-known models in the macroeconomic literature. This book is motivated by my strong desire to persuade some traditionally trained economists to phrase their questions in stochastic ways and apply some of the methods presented in it to their work.

Mainstream economists and graduate students of economics may wonder why to use stochastic models or what additional or new insights they yield or, if stochastic laws in economics are so useful, why they have not heard of them before. A short answer is that models with finite numbers of agents in appropriate stochastic contexts reveal interesting economic phenomena that are invisible in deterministic models with infinite numbers of (representative) agents. Traditional models wash out some important information about economies, but one would not know them. This finitary and stochastic approach provides more information about the economy than deterministic economic laws permit.

There are many areas of economics to which my approach applies. In speaking of inflation and unemployment, Tobin, in his presidential address at the American Economic Association Meeting in 1971, came close to describing stochastic laws and aggregate dynamics and fluctuations (in terms of Fokker–Planck equations, say), according to my way of modeling, when he said, “. . . stochastic macro-equilibrium, stochastic, because random intersectoral shocks keep individual labor markets in diverse states of disequilibrium;

macro-equilibrium, because the perpetual flux of particular markets produces fairly definite aggregate outcomes of unemployment and wages . . . .”

Another major class of examples is building business-cycle models. All kinds of theories are found in the literature, and new theories keep cropping up. The real business cycle (RBC) theory by Kydland and Prescott (1982) may arguably be the most influential current theory among mainstream economists. As typified by the RBC, a natural research strategy to study business cycles is to explain fluctuations as a direct outcome of the behavior of *individual agents*. The more strongly one wishes to interpret aggregate fluctuations as something “rational” or “optimal,” the more one is led to this essentially microeconomic approach. The mission of this approach is to explain fluctuations as responses of individual agents to changes in their economic environments. The consumer’s intertemporal substitution, for example, is a device to achieve this goal. This has been the standard approach in the mainstream economics in the last twenty years or so.

Surely, we would like to know the distribution of durations of “good” times and “bad” times. When models admit multiple equilibria, which equilibrium, if any, will the model settle in? How long can the system be expected to stay in one basin of attraction before it moves to another? And so on. This book presents a different approach to fluctuations. This alternative approach is based on the fact that economy consists of a large number of agents or sectors. (The population of a large industrialized economy, for example, contains of the order of  $10^8$  agents.) Even if agents intertemporally maximize their respective objective functions, their environments or constraints all differ and are always subject to idiosyncratic shocks. Our alternative approach emphasizes that an outcome of interactions of a large number of agents facing such incessant idiosyncratic shocks cannot be described by a response of the representative agent and calls for a model of stochastic processes. In a seminal work, Slutsky (1937) proposed a stochastic approach. We follow his lead in this book to build a stochastic model of fluctuations and growth.

Although studies of macroeconomy with many heterogeneous agents are not new, dynamic behavior of economies in disequilibrium is not satisfactorily analyzed. The traditional Walrasian economy is the egregious example. It focuses on price adjustment with the help of nonexistent auctioneer.

In a nutshell, this book formulates and analyzes a large but finite number of interacting economic agents as continuous-time Markov chains with discrete state spaces. Dynamics are described in terms of the backward Chapman–Kolmogorov equations, also known as the master equations. We are interested in such questions as the existence of stationary probability distributions for some variables, of critical points of aggregate dynamics, and of fluctuations about locally stable equilibria, and in the distributions of relative sizes of the basins of attraction and associated probabilities, how they relate to the lengths

## Preface

xv

of business cycles, and so forth. The agents are assumed to be exchangeable rather than representative and have either a finite or countably infinite number of decisions to choose from, or they belong to a finite or countably infinite number of types or categories.

Unlike the jump Markov processes treated in standard textbooks on probability or stochastic processes, the transition rates of the processes in this book are endogenously determined via the value maximizations by the agents in the model. Using this framework, we take fresh looks at some well-known search models, such as the Diamond model and disequilibrium quantity adjustment models, as well as models for diffusions of innovations and endogenous growth. Formulations involving a few large clusters of agents in markets, and the implications for the volatility of returns on financial-asset markets, which may develop from interaction of many agents, are also examined using random combinatorial analysis. Such investigations lead to results not usually discussed in the traditional macroeconomic literature, such as the existence of power laws for some variables of interest, discoveries that some common laws apply to seemingly unrelated areas, and so on.

This book is aimed at advanced graduate students and practicing professionals in economics, as well as in some related areas, such as the recently formed area of econophysics. Some of the topics have been discussed in my graduate courses at UCLA and at Keio University, Tokyo, and at several conferences, workshops, and seminars. I wish to express appreciation to Professors R. Craine, K. Kawamata, A. Kirman, M. Marchesi, T. Lux, W. Semmler, H. Yoshikawa, and J.-B. Zimmermann for opportunities for presenting talks, and to Professors Y. Shirai, D. Costantini, U. Garibaldi, and D. Sornette for their useful comments on some parts of the topics in the book. I am particularly indebted to Professors Yoshikawa, Costantini, and Garibaldi for their help and guidance in overcoming my ignorance and misunderstandings. Simulations reported in this book were programmed by a former and a current graduate student at UCLA, J. Nagamine and R. Singh. I thank them for their help.

Some of my research activities reported here were supported in part by grants from the Academic Senate of the University of California, Los Angeles.